Second Order Linear Differential Equations

4. The auxiliary equation is $2r^2 - r - 1 = (2r+1)(r-1) = 0 \Rightarrow r = 1, r = -\frac{1}{2}$. Then the general solution is $y = c_1 e^x + c_2 e^{-x/2}$.

6. The auxiliary equation is $3r^2 - 5r = r(3r-5) = 0 \Rightarrow r = 0, r = \frac{5}{3}$, so $y = c_1 + c_2 e^{5x/3}$.

8. The auxiliary equation is $16r^2 + 24r + 9 = (4r+3)^2 = 0 \Rightarrow r = -\frac{3}{4}$, so $y = c_1 e^{-3x/4} + c_2 x e^{-3x/4}$.

18. $r^2 + 3 = 0 \Rightarrow r = \pm\sqrt{3}i$ and the general solution is $y = e^{0x}(c_1 \cos(\sqrt{3}\,x) + c_2 \sin(\sqrt{3}\,x)) = c_1 \cos(\sqrt{3}\,x) + c_2 \sin(\sqrt{3}\,x)$. Then $y(0) = 1 \Rightarrow c_1 = 1$ and $y'(0) = 3 \Rightarrow c_2 = \sqrt{3}$, so the solution to the initial-value problem is $y = \cos(\sqrt{3}\,x) + \sqrt{3}\sin(\sqrt{3}\,x)$.

20. $2r^2 + 5r - 3 = (2r-1)(r+3) = 0 \Rightarrow r = \frac{1}{2}, r = -3$ and the general solution is $y = c_1 e^{x/2} + c_2 e^{-3x}$. Then $1 = y(0) = c_1 + c_2$ and $4 = y'(0) = \frac{1}{2}c_1 - 3c_2$ so $c_1 = 2$, $c_2 = -1$ and the solution to the initial-value problem is $y = 2e^{x/2} - e^{-3x}$.

26. $r^2 + 2r = r(2+r) = 0 \Rightarrow r = 0, r = -2$ and the general solution is $y = c_1 + c_2 e^{-2x}$. Then $1 = y(0) = c_1 + c_2$ and $2 = y(1) = c_1 + c_2 e^{-2}$ so $c_2 = \dfrac{e^2}{1-e^2}$, $c_1 = \dfrac{1-2e^2}{1-e^2}$. The solution of the boundary-value problem is $y = \dfrac{1-2e^2}{1-e^2} + \dfrac{e^2}{1-e^2} \cdot e^{-2x}$.

28. $r^2 + 100 = 0 \Rightarrow r = \pm 10i$ and the general solution is $y = c_1 \cos 10x + c_2 \sin 10x$. But $2 = y(0) = c_1$ and $5 = y(\pi) = c_1$, so there is no solution.

30. $r^2 - 6r + 9 = (r-3)^2 = 0 \Rightarrow r = 3$ and the general solution is $y = c_1 e^{3x} + c_2 x e^{3x}$. Then $1 = y(0) = c_1$ and $0 = y(1) = c_1 e^3 + c_2 e^3 \Rightarrow c_2 = -1$. The solution of the boundary-value problem is $y = e^{3x} - xe^{3x}$.

34. The auxiliary equation is $ar^2 + br + c = 0$. If $b^2 - 4ac > 0$, then any solution is of the form $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$ where $r_1 = \dfrac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $r_2 = \dfrac{-b - \sqrt{b^2 - 4ac}}{2a}$. But a, b, and c are all positive so both r_1 and r_2 are negative and $\lim_{x\to\infty} y(x) = 0$. If $b^2 - 4ac = 0$, then any solution is of the form $y(x) = c_1 e^{rx} + c_2 x e^{rx}$ where $r = -b/(2a) < 0$ since a, b are positive. Hence $\lim_{x\to\infty} y(x) = 0$. Finally if $b^2 - 4ac < 0$, then any solution is of the form $y(x) = e^{\alpha x}(c_1 \cos\beta x + c_2 \sin\beta x)$ where $\alpha = -b/(2a) < 0$ since a and b are positive. Thus $\lim_{x\to\infty} y(x) = 0$.

Using Series to Solve Differential Equations

2. Let $y(x) = \sum\limits_{n=0}^{\infty} c_n x^n$. Then $y' = xy \Rightarrow y' - xy = 0 \Rightarrow \sum\limits_{n=1}^{\infty} nc_n x^{n-1} - x\sum\limits_{n=0}^{\infty} c_n x^n = 0$ or $\sum\limits_{n=1}^{\infty} nc_n x^{n-1} - \sum\limits_{n=0}^{\infty} c_n x^{n+1} = 0$. Replacing n with $n+1$ in the first sum and n with $n-1$ in the second gives $\sum\limits_{n=0}^{\infty} (n+1)c_{n+1}x^n - \sum\limits_{n=1}^{\infty} c_{n-1}x^n = 0$ or $c_1 + \sum\limits_{n=1}^{\infty} (n+1)c_{n+1}x^n - \sum\limits_{n=1}^{\infty} c_{n-1}x^n = 0$. Thus,

$c_1 + \sum_{n=1}^{\infty} [(n+1)c_{n+1} - c_{n-1}]\, x^n = 0$. Equating coefficients gives $c_1 = 0$ and $(n+1)\,c_{n+1} - c_{n-1} = 0$. Thus, the recursion relation is $c_{n+1} = \dfrac{c_{n-1}}{n+1}$, $n = 1, 2, \ldots$. But $c_1 = 0$, so $c_3 = 0$ and $c_5 = 0$ and in general $c_{2n+1} = 0$. Also, $c_2 = \dfrac{c_0}{2}$, $c_4 = \dfrac{c_2}{4} = \dfrac{c_0}{4 \cdot 2} = \dfrac{c_0}{2^2 \cdot 2!}$, $c_6 = \dfrac{c_4}{6} = \dfrac{c_0}{6 \cdot 4 \cdot 2} = \dfrac{c_0}{2^3 \cdot 3!}$ and in general $c_{2n} = \dfrac{c_0}{2^n \cdot n!}$. Thus, the solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{2n} x^{2n} = \sum_{n=0}^{\infty} \frac{c_0}{2^n \cdot n!} x^{2n} = c_0 \sum_{n=0}^{\infty} \frac{(x^2/2)^n}{n!} = c_0 e^{x^2/2}$$

4. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n \quad \Rightarrow \quad y'(x) = \sum_{n=1}^{\infty} n c_n x^{n-1} = \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n$. Then the differential equation becomes $(x-3) \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n + 2 \sum_{n=0}^{\infty} c_n x^n = 0 \;\Rightarrow$

$\sum_{n=0}^{\infty} (n+1) c_{n+1} x^{n+1} - 3 \sum_{n=0}^{\infty} (n+1) c_{n+1} x^n + 2 \sum_{n=0}^{\infty} c_n x^n = 0 \;\Rightarrow$

$\sum_{n=1}^{\infty} n c_n x^n - \sum_{n=0}^{\infty} 3(n+1) c_{n+1} x^n + \sum_{n=0}^{\infty} 2 c_n x^n = 0 \;\Rightarrow\; \sum_{n=0}^{\infty} [(n+2) c_n - 3(n+1) c_{n+1}]\, x^n = 0$

(since $\sum_{n=1}^{\infty} n c_n x^n = \sum_{n=0}^{\infty} n c_n x^n$). Equating coefficients gives $(n+2) c_n - 3(n+1) c_{n+1} = 0$, thus the recursion relation is $c_{n+1} = \dfrac{(n+2) c_n}{3(n+1)}$, $n = 0, 1, 2, \ldots$. Then $c_1 = \dfrac{2c_0}{3}$, $c_2 = \dfrac{3c_1}{3(2)} = \dfrac{3c_0}{3^2}$, $c_3 = \dfrac{4c_2}{3(3)} = \dfrac{4c_0}{3^3}$, $c_4 = \dfrac{5c_3}{3(4)} = \dfrac{5c_0}{3^4}$, and in general, $c_n = \dfrac{(n+1)c_0}{3^n}$. Thus the solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 \sum_{n=0}^{\infty} \frac{n+1}{3^n} x^n. \left[\text{Note that } c_0 \sum_{n=0}^{\infty} \frac{n+1}{3^n} x^n = \frac{9c_0}{(3-x)^2} \text{ for } |x| < 3.\right]$$

6. Let $y(x) = \sum_{n=0}^{\infty} c_n x^n$. Then $y''(x) = \sum_{n=2}^{\infty} n(n-1) c_n x^{n-2} = \sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n$. Hence, the equation $y'' = y$ becomes $\sum_{n=0}^{\infty} (n+2)(n+1) c_{n+2} x^n - \sum_{n=0}^{\infty} c_n x^n = 0$ or $\sum_{n=0}^{\infty} [(n+2)(n+1) c_{n+2} - c_n] x^n = 0$. So the recursion relation is $c_{n+2} = \dfrac{c_n}{(n+2)(n+1)}$, $n = 0, 1, \ldots$. Given c_0 and c_1, $c_2 = \dfrac{c_0}{2 \cdot 1}$, $c_4 = \dfrac{c_2}{4 \cdot 3} = \dfrac{c_0}{4!}$, $c_6 = \dfrac{c_4}{6 \cdot 5} = \dfrac{c_0}{6!}, \ldots, c_{2n} = \dfrac{c_0}{(2n)!}$ and $c_3 = \dfrac{c_1}{3 \cdot 2}$, $c_5 = \dfrac{c_3}{5 \cdot 4} = \dfrac{c_1}{5 \cdot 4 \cdot 3 \cdot 2} = \dfrac{c_1}{5!}$, $c_7 = \dfrac{c_5}{7 \cdot 6} = \dfrac{c_1}{7!}, \ldots,$ $c_{2n+1} = \dfrac{c_1}{(2n+1)!}$. Thus, the solution is

$$y(x) = \sum_{n=0}^{\infty} c_n x^n = \sum_{n=0}^{\infty} c_{2n} x^{2n} + \sum_{n=0}^{\infty} c_{2n+1} x^{2n+1} = c_0 \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!} + c_1 \sum_{n=0}^{\infty} \frac{x^{2n+1}}{(2n+1)!}$$

The solution can be written as $y(x) = c_0 \cosh x + c_1 \sinh x$

$$\left[\text{or } y(x) = c_0 \frac{e^x + e^{-x}}{2} + c_1 \frac{e^x - e^{-x}}{2} = \frac{c_0 + c_1}{2} e^x + \frac{c_0 - c_1}{2} e^{-x}\right].$$

8. Assuming $y(x)=\sum_{n=0}^{\infty}c_nx^n$, $y''(x)=\sum_{n=2}^{\infty}n(n-1)c_nx^{n-2}=\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n$ and

$-xy(x)=-\sum_{n=0}^{\infty}c_nx^{n+1}=-\sum_{n=1}^{\infty}c_{n-1}x^n$. The equation $y''=xy$ becomes

$\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^n-\sum_{n=1}^{\infty}c_{n-1}x^n=0$ or $2c_2+\sum_{n=1}^{\infty}[(n+2)(n+1)c_{n+2}-c_{n-1}]x^n=0$. Equating

coefficients gives $c_2=0$ and $c_{n+2}=\dfrac{c_{n-1}}{(n+2)(n+1)}$ for $n=1,2,\ldots$. Since $c_2=0$,

$c_{3n+2}=0$ for $n=0,1,2,\ldots$. Given c_0, $c_3=\dfrac{c_0}{3\cdot 2}$, $c_6=\dfrac{c_3}{6\cdot 5}=\dfrac{c_0}{6\cdot 5\cdot 3\cdot 2},\ldots,$

$c_{3n}=\dfrac{c_0}{3n(3n-1)(3n-3)(3n-4)\cdot\cdots\cdot 6\cdot 5\cdot 3\cdot 2}$. Given c_1, $c_4=\dfrac{c_1}{4\cdot 3}$, $c_7=\dfrac{c_4}{7\cdot 6}=\dfrac{c_1}{7\cdot 6\cdot 4\cdot 3},\ldots,$

$c_{3n+1}=\dfrac{c_1}{(3n+1)3n(3n-2)(3n-3)\ldots 7\cdot 6\cdot 4\cdot 3}$. The solution can be written as

$$y(x)=c_0\sum_{n=0}^{\infty}\frac{(3n-2)(3n-5)\cdot\cdots\cdot 7\cdot 4\cdot 1}{(3n)!}x^{3n}+c_1\sum_{n=0}^{\infty}\frac{(3n-1)(3n-4)\cdot\cdots\cdot 8\cdot 5\cdot 2}{(3n+1)!}x^{3n+1}$$

10. Assuming that $y(x)=\sum_{n=0}^{\infty}c_nx^n$, we have $x^2y=\sum_{n=0}^{\infty}c_nx^{n+2}$ and

$$\begin{aligned}y''(x)&=\sum_{n=2}^{\infty}n(n-1)c_nx^{n-2}=\sum_{n=-2}^{\infty}(n+4)(n+3)c_{n+4}x^{n+2}\\&=2c_2+6c_3x+\sum_{n=0}^{\infty}(n+4)(n+3)c_{n+4}x^{n+2}\end{aligned}$$

Thus, the equation $y''+x^2y=0$ becomes $2c_2+6c_3x+\sum_{n=0}^{\infty}[(n+4)(n+3)c_{n+4}+c_n]\,x^{n+2}=0$. So

$c_2=c_3=0$ and the recursion relation is $c_{n+4}=-\dfrac{c_n}{(n+4)(n+3)}$, $n=0,1,2,\ldots$.

But $c_1=y'(0)=0=c_2=c_3$ and by the recursion relation, $c_{4n+1}=c_{4n+2}=c_{4n+3}=0$ for $n=0,1,2,\ldots$.
Also, $c_0=y(0)=1$, so

$c_4=-\dfrac{c_0}{4\cdot 3}=-\dfrac{1}{4\cdot 3}$, $c_8=-\dfrac{c_4}{8\cdot 7}=\dfrac{(-1)^2}{8\cdot 7\cdot 4\cdot 3},\ldots,c_{4n}=\dfrac{(-1)^n}{4n(4n-1)(4n-4)(4n-5)\cdot\cdots\cdot 4\cdot 3}$.

Thus, the solution to the initial-value problem is

$$y(x)=\sum_{n=0}^{\infty}c_nx^n=c_0+\sum_{n=0}^{\infty}c_{4n}x^{4n}=1+\sum_{n=1}^{\infty}(-1)^n\frac{x^{4n}}{4n(4n-1)(4n-4)(4n-5)\cdot\cdots\cdot 4\cdot 3}$$

12. (a) Let $y(x)=\sum_{n=0}^{\infty}c_nx^n$. Then $x^2y''(x)=\sum_{n=2}^{\infty}n(n-1)c_nx^n=\sum_{n=0}^{\infty}(n+2)(n+1)c_{n+2}x^{n+2}$,

$xy'(x)=\sum_{n=1}^{\infty}nc_nx^n=\sum_{n=-1}^{\infty}(n+2)c_{n+2}x^{n+2}=c_1x+\sum_{n=0}^{\infty}(n+2)c_{n+2}x^{n+2}$, and the equation

$x^2y''+xy'+x^2y=0$ becomes $c_1x+\sum_{n=0}^{\infty}\{[(n+2)(n+1)+(n+2)]c_{n+2}+c_n\}x^{n+2}=0$. So $c_1=0$

and the recursion relation is $c_{n+2}=-\dfrac{c_n}{(n+2)^2}$, $n=0,1,2,\ldots$. But $c_1=y'(0)=0$ so $c_{2n+1}=0$ for

$n=0,1,2,\ldots$. Also, $c_0=y(0)=1$, so $c_2=-\dfrac{1}{2^2}$, $c_4=-\dfrac{c_2}{4^2}=(-1)^2\dfrac{1}{4^22^2}=(-1)^2\dfrac{1}{2^4(2!)^2}$,

$c_6=-\dfrac{c_4}{6^2}=(-1)^3\dfrac{1}{2^6(3!)^2},\ldots,c_{2n}=(-1)^n\dfrac{1}{2^{2n}(n!)^2}$. The solution is

$$y(x)=\sum_{n=0}^{\infty}c_nx^n=\sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{2^{2n}(n!)^2}$$