2. (a) Given the power series $\sum_{n=0}^{\infty} c_n(x-a)^n$, the radius of convergence is:

(i) 0 if the series converges only when $x=a$

(ii) ∞ if the series converges for all x, or

(iii) a positive number R such that the series converges if $|x-a|<R$ and diverges if $|x-a|>R$.

In most cases, R can be found by using the Ratio Test.

(b) The interval of convergence of a power series is the interval that consists of all values of x for which the series converges. Corresponding to the cases in part (a), the interval of convergence is: (i) the single point $\{a\}$, (ii) all real numbers; that is, the real number line $(-\infty,\infty)$, or (iii) an interval with endpoints $a-R$ and $a+R$ which can contain neither, either, or both of the endpoints. In this case, we must test the series for convergence at each endpoint to determine the interval of convergence.

4. If $a_n = \dfrac{(-1)^n x^n}{n+1}$, then $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{n+2}\cdot\dfrac{n+1}{x^n}\right| = \lim\limits_{n\to\infty}\dfrac{|x|}{1+1/(n+1)} = |x|$. By the Ratio Test, the series $\sum\limits_{n=0}^{\infty}\dfrac{(-1)^n x^n}{n+1}$ converges when $|x|<1$, so $R=1$. When $x=-1$, the series diverges because it is the harmonic series; when $x=1$, it is the alternating harmonic series, which converges by the Alternating Series Test. Thus, $I=(-1,1]$.

6. $a_n = \sqrt{n}\,x^n$, so we need $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\dfrac{\sqrt{n+1}\,|x|^{n+1}}{\sqrt{n}\,|x|^n} = \lim\limits_{n\to\infty}\sqrt{1+\dfrac{1}{n}}\,|x| = |x| < 1$ for convergence (by the Ratio Test), so $R=1$. When $x=\pm1$, $\lim\limits_{n\to\infty}|a_n| = \lim\limits_{n\to\infty}\sqrt{n}=\infty$, so the series diverges by the Test for Divergence. Thus, $I=(-1,1)$.

10. $a_n = \dfrac{x^n}{5^n n^5}$, so $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\left|\dfrac{x^{n+1}}{5^{n+1}(n+1)^5}\cdot\dfrac{5^n n^5}{x^n}\right| = \lim\limits_{n\to\infty}\dfrac{|x|}{5}\left(\dfrac{n}{n+1}\right)^5 = \dfrac{|x|}{5}$. By the Ratio Test, the series $\sum\limits_{n=0}^{\infty}\dfrac{x^n}{5^n n^5}$ converges when $\dfrac{|x|}{5}<1 \quad\Leftrightarrow\quad |x|<5$, so $R=5$. When $x=-5$, we get the series $\sum\limits_{n=1}^{\infty}\dfrac{(-1)^n}{n^5}$, which converges by the Alternating Series Test. When $x=5$, we get the convergent p-series $\sum\limits_{n=1}^{\infty}\dfrac{1}{n^5}\quad[p=5>1]$. Thus, $I=[-5,5]$.

12. $a_n = (-1)^n\dfrac{x^{2n}}{(2n)!}$, so $\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\dfrac{|x|^{2n+2}}{(2n+2)!}\cdot\dfrac{(2n)!}{|x|^{2n}} = \lim\limits_{n\to\infty}\dfrac{|x|^2}{(2n+1)(2n+2)} = 0$. Thus, by the Ratio Test, the series converges for *all* real x and we have $R=\infty$ and $I=(-\infty,\infty)$.

18. $a_n = \dfrac{n^2x^n}{2\cdot4\cdot6\cdot\dots\cdot(2n)} = \dfrac{n^2x^n}{2^n n!} = \dfrac{nx^n}{2^n(n-1)!}$, so

$\lim\limits_{n\to\infty}\left|\dfrac{a_{n+1}}{a_n}\right| = \lim\limits_{n\to\infty}\dfrac{(n+1)\,|x|^{n+1}}{2^{n+1}n!}\cdot\dfrac{2^n(n-1)!}{n\,|x|^n} = \lim\limits_{n\to\infty}\dfrac{n+1}{n^2}\dfrac{|x|}{2} = 0$. Thus, by the Ratio Test, the series converges for *all* real x and we have $R=\infty$ and $I=(-\infty,\infty)$.

20. We are given that the power series $\sum_{n=0}^{\infty} c_n x^n$ is convergent for $x = -4$ and divergent when $x = 6$. So by Theorem 3 it converges for at least $-4 \le x < 4$ and diverges for at least $x \ge 6$ and $x < -6$. Therefore:

(a) It converges when $x = 1$; that is, $\sum c_n$ is convergent.

(b) It diverges when $x = 8$; that is, $\sum c_n 8^n$ is divergent.

(c) It converges when $x = -3$; that is, $\sum c_n(-3^n)$ is convergent.

(d) It diverges when $x = -9$; that is, $\sum c_n(-9)^n = \sum(-1)^n c_n 9^n$ is divergent.

2. If $f(x) = \sum_{n=0}^{\infty} b_n x^n$ converges on $(-2, 2)$, then $\int f(x)dx = C + \sum_{n=0}^{\infty} \frac{b_n}{n+1} x^{n+1}$ has the same radius of convergence (by Theorem 2), but may not have the same interval of convergence—it may happen that the integrated series converges at an endpoint (or both endpoints).

4. $f(x) = \frac{3}{1-x^4} = 3\left(\frac{1}{1-x^4}\right) = 3(1 + x^4 + x^8 + x^{12} + \cdots) = 3\sum_{n=0}^{\infty}(x^4)^n = \sum_{n=0}^{\infty} 3x^{4n}$ with $|x^4| < 1 \quad \Leftrightarrow$ $|x| < 1$, so $R = 1$ and $I = (-1, 1)$.

$\left[\text{Note that } 3\sum_{n=0}^{\infty}(x^4)^n \text{ converges} \quad \Leftrightarrow \quad \sum_{n=0}^{\infty}(x^4)^n \text{ converges, so the appropriate condition [from Equation (1)] is } |x^4| < 1.\right]$

10. $f(x) = \frac{x^2}{a^3 - x^3} = \frac{x^2}{a^3} \cdot \frac{1}{1 - x^3/a^3} = \frac{x^2}{a^3}\sum_{n=0}^{\infty}\left(\frac{x^3}{a^3}\right)^n = \sum_{n=0}^{\infty}\frac{x^{3n+2}}{a^{3n+3}}$. The series converges when $|x^3/a^3| < 1 \quad \Leftrightarrow$ $|x^3| < |a^3| \quad \Leftrightarrow \quad |x| < |a|$, so $R = |a|$ and $I = (-|a|, |a|)$.

12. $f(x) = \frac{7x-1}{3x^2+2x-1} = \frac{7x-1}{(3x-1)(x+1)} = \frac{A}{3x-1} + \frac{B}{x+1} = \frac{1}{3x-1} + \frac{2}{x+1} = 2 \cdot \frac{1}{1-(-x)} - \frac{1}{1-3x}$

$$= 2\sum_{n=0}^{\infty}(-x)^n - \sum_{n=0}^{\infty}(3x)^n = \sum_{n=0}^{\infty}[2(-1)^n - 3^n]x^n$$

The series $\sum(-x)^n$ converges for $x \in (-1, 1)$ and the series $\sum(3x)^n$ converges for $x \in \left(-\frac{1}{3}, \frac{1}{3}\right)$, so their sum converges for $x \in \left(-\frac{1}{3}, \frac{1}{3}\right) = I$.

14. (a) $\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-1)^n x^n$ [geometric series with $R = 1$], so

$$f(x) = \ln(1+x) = \int \frac{dx}{1+x} = \int \left[\sum_{n=0}^{\infty}(-1)^n x^n\right] dx = C + \sum_{n=0}^{\infty}(-1)^n \frac{x^{n+1}}{n+1} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}$$

$[C = 0 \text{ since } f(0) = \ln 1 = 0]$, with $R = 1$

(b) $f(x) = x\ln(1+x) = x\left[\sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^n}{n}\right]$ [by part (a)] $= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{n+1}}{n} = \sum_{n=2}^{\infty}\frac{(-1)^n x^n}{n-1}$ with $R = 1$.

(c) $f(x) = \ln(x^2+1) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}(x^2)^n}{n}$ [by part (a)] $= \sum_{n=1}^{\infty}\frac{(-1)^{n-1}x^{2n}}{n}$ with $R = 1$.

16. We know that $\frac{1}{1-2x} = \sum_{n=0}^{\infty} (2x)^n$. Differentiating, we get $\frac{2}{(1-2x)^2} = \sum_{n=1}^{\infty} 2^n n x^{n-1} = \sum_{n=0}^{\infty} 2^{n+1}(n+1)x^n$, so

$$f(x) = \frac{x^2}{(1-2x)^2} = \frac{x^2}{2} \cdot \frac{2}{(1-2x)^2} = \frac{x^2}{2} \sum_{n=0}^{\infty} 2^{n+1}(n+1)x^n = \sum_{n=0}^{\infty} 2^n (n+1) x^{n+2} \text{ or } \sum_{n=2}^{\infty} 2^{n-2}(n-1)x^n,$$

with $R = \frac{1}{2}$.

26. By Example 7, $\int \tan^{-1}(x^2)\,dx = \int \sum_{n=0}^{\infty} (-1)^n \frac{(x^2)^{2n+1}}{2n+1}\,dx = C + \sum_{n=0}^{\infty} (-1)^n \frac{x^{4n+3}}{(2n+1)(4n+3)}$ with $R = 1$.

28. From Example 6, we know $\ln(1-x) = -\sum_{n=1}^{\infty} \frac{x^n}{n}$, so

$$\ln(1+x^4) = \ln[1-(-x^4)] = -\sum_{n=1}^{\infty} \frac{(-x^4)^n}{n} = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{4n}}{n} \quad \Rightarrow$$

$$\int \ln(1+x^4)\,dx = \int \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{4n}}{n}\,dx = C + \sum_{n=1}^{\infty} (-1)^{n+1} \frac{x^{4n+1}}{n(4n+1)}. \text{ Thus,}$$

$$I = \int_0^{0.4} \ln(1+x^4)\,dx = \left[\frac{x^5}{5} - \frac{x^9}{18} + \frac{x^{13}}{39} - \frac{x^{17}}{68} + \cdots\right]_0^{0.4} = \frac{(0.4)^5}{5} - \frac{(0.4)^9}{18} + \frac{(0.4)^{13}}{39} - \frac{(0.4)^{17}}{68} + \cdots.$$

The series is alternating, so if we use the first three terms, the error is at most $(0.4)^{17}/68 \approx 2.5 \times 10^{-9}$.

So $I \approx (0.4)^5/5 - (0.4)^9/18 + (0.9)^{13}/39 \approx 0.002034$ to six decimal places.

32. $f(x) = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} \quad \Rightarrow \quad f'(x) = \sum_{n=1}^{\infty} \frac{(-1)^n 2n x^{2n-1}}{(2n)!}$ [the first term disappears], so

$$f''(x) = \sum_{n=1}^{\infty} \frac{(-1)^n (2n)(2n-1) x^{2n-2}}{(2n)!} = \sum_{n=1}^{\infty} \frac{(-1)^n x^{2(n-1)}}{[2(n-1)]!} = \sum_{n=0}^{\infty} \frac{(-1)^{n+1} x^{2n}}{(2n)!} \quad \text{[substituting } n+1 \text{ for } n\text{]}$$

$$= -\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = -f(x) \quad \Rightarrow \quad f''(x) + f(x) = 0.$$

2. (a) Using Formula 6, a power series expansion of f at 1 must have the form $f(1) + f'(1)(x-1) + \cdots$. Comparing to the given series, $1.6 - 0.8(x-1) + \cdots$, we must have $f'(1) = -0.8$. But from the graph, $f'(1)$ is positive. Hence, the given series is *not* the Taylor series of f centered at 1.

(b) A power series expansion of f at 2 must have the form $f(2) + f'(2)(x-2) + \frac{1}{2}f''(2)(x-2)^2 + \cdots$. Comparing to the given series, $2.8 + 0.5(x-2) + 1.5(x-2)^2 - 0.1(x-2)^3 + \cdots$, we must have $\frac{1}{2}f''(2) = 1.5$; that is, $f''(2)$ is positive. But from the graph, f is concave downward near $x = 2$, so $f''(2)$ must be negative. Hence, the given series is *not* the Taylor series of f centered at 2.

4. Since $f^{(n)}(4) = \dfrac{(-1)^n\, n!}{3^n(n+1)}$, Equation 6 gives the Taylor series

$\displaystyle\sum_{n=0}^{\infty} \frac{f^{(n)}(4)}{n!}(x-4)^n = \sum_{n=0}^{\infty} \frac{(-1)^n\, n!}{3^n(n+1)\, n!}(x-4)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{3^n(n+1)}(x-4)^n$, which is the Taylor series for f centered at 4. Apply the Ratio Test to find the radius of convergence R.

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}(x-4)^{n+1}}{3^{n+1}(n+2)} \cdot \frac{3^n(n+1)}{(-1)^n(x-4)^n}\right| = \lim_{n\to\infty}\left|\frac{(-1)(x-4)(n+1)}{3(n+2)}\right|$$

$$= \frac{1}{3}|x-4| \lim_{n\to\infty} \frac{n+1}{n+2} = \frac{1}{3}|x-4|$$

For convergence, $\frac{1}{3}|x-4| < 1 \quad \Leftrightarrow \quad |x-4| < 3$, so $R = 3$.

8.

n	$f^{(n)}(x)$	$f^{(n)}(0)$
0	xe^x	0
1	$(x+1)e^x$	1
2	$(x+2)e^x$	2
3	$(x+3)e^x$	3
$\vdots$	$\vdots$	$\vdots$

$$xe^x = \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n = \sum_{n=0}^{\infty} \frac{n}{n!}x^n = \sum_{n=1}^{\infty} \frac{n}{n!}x^n = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!}.$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{|x|^{n+1}}{n!} \cdot \frac{(n-1)!}{|x|^n}\right] = \lim_{n\to\infty} \frac{|x|}{n}$$

$$= 0 < 1 \text{ for all } x$$

So $R = \infty$.

14.

n	$f^{(n)}(x)$	$f^{(n)}(2)$
0	$\ln x$	$\ln 2$
1	x^{-1}	$\frac{1}{2}$
2	$-x^{-2}$	$-\frac{1}{4}$
3	$2x^{-3}$	$\frac{2}{8}$
4	$-3 \cdot 2x^{-4}$	$-\frac{3 \cdot 2}{16}$
$\vdots$	$\vdots$	$\vdots$

$f^{(n)}(2) = \dfrac{(-1)^{n-1}(n-1)!}{2^n}$ for $n \geq 1$, so

$$\ln x = \ln 2 + \sum_{n=1}^{\infty} \frac{(-1)^{n-1}(x-2)^n}{n \cdot 2^n}.$$

$\displaystyle\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \frac{|x-2|}{2} \lim_{n\to\infty} \frac{n}{n+1} = \frac{|x-2|}{2} < 1$ for convergence,

so $|x-2| < 2 \quad \Rightarrow \quad R = 2$.

16.

n	$f^{(n)}(x)$	$f^{(n)}(\pi/2)$
0	$\sin x$	1
1	$\cos x$	0
2	$-\sin x$	-1
3	$-\cos x$	0
4	$\sin x$	1
$\vdots$	$\vdots$	$\vdots$

$$\sin x = \sum_{k=0}^{\infty} \frac{f^{(k)}(\pi/2)}{k!}\left(x-\frac{\pi}{2}\right)^k = 1 - \frac{(x-\pi/2)^2}{2!} + \frac{(x-\pi/2)^4}{4!} - \frac{(x-\pi/2)^6}{6!} + \cdots = \sum_{n=0}^{\infty} (-1)^n \frac{(x-\pi/2)^{2n}}{(2n)!}$$

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left[\frac{|x-\pi/2|^{2n+2}}{(2n+2)!}\cdot\frac{(2n)!}{|x-\pi/2|^{2n}}\right] = \lim_{n\to\infty}\frac{|x-\pi/2|^2}{(2n+2)(2n+1)} = 0 < 1 \text{ for all } x$$

So $R = \infty$.

20. If $f(x) = \sin x$, then by Taylor's Formula $R_n(x) = \dfrac{f^{(n+1)}(z)}{(n+1)!}\left(x-\dfrac{\pi}{2}\right)^{n+1}$, where z lies between x and $\pi/2$. But $f^{(n+1)}(z) = \pm\sin z$ or $\pm\cos z$. In each case, $\left|f^{(n+1)}(z)\right| \le 1$, so $|R_n(x)| \le \dfrac{1}{(n+1)!}\left|x-\dfrac{\pi}{2}\right|^{n+1}$. Thus, $|R_n(x)| \to 0$ as $n\to\infty$ by Equation 11. So $\lim_{n\to\infty} R_n(x) = 0$ and, by Theorem 8, the series in Exercise 16 represents $\sin x$ for all x.

24. $\dfrac{1}{(1+x)^4} = (1+x)^{-4} = \displaystyle\sum_{n=0}^{\infty}\binom{-4}{n}x^n$. The binomial coefficient is

$$\binom{-4}{n} = \frac{(-4)(-5)(-6)\cdot\cdots\cdot(-4-n+1)}{n!} = \frac{(-4)(-5)(-6)\cdot\cdots\cdot[-(n+3)]}{n!}$$

$$= \frac{(-1)^n\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot\cdots\cdot(n+1)(n+2)(n+3)}{2\cdot 3\cdot n!} = \frac{(-1)^n(n+1)(n+2)(n+3)}{6}$$

Thus, $\dfrac{1}{(1+x)^4} = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n(n+1)(n+2)(n+3)}{6}x^n$ for $|x|<1$, so $R = 1$.

28. $e^x = \displaystyle\sum_{n=0}^{\infty}\frac{x^n}{n!} \quad\Rightarrow\quad f(x) = e^{-x/2} = \sum_{n=0}^{\infty}\frac{(-x/2)^n}{n!} = \sum_{n=0}^{\infty}\frac{(-1)^n}{2^n\,n!}x^n, \quad R = \infty$

34. $$\frac{x^2}{\sqrt{2+x}} = \frac{x^2}{\sqrt{2\,(1+x/2)}} = \frac{x^2}{\sqrt{2}}\left(1+\frac{x}{2}\right)^{-1/2} = \frac{x^2}{\sqrt{2}}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\frac{x}{2}\right)^n$$

$$= \frac{x^2}{\sqrt{2}}\left[1+\left(-\tfrac{1}{2}\right)\left(\frac{x}{2}\right)+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)}{2!}\left(\frac{x}{2}\right)^2+\frac{\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)\left(-\frac{5}{2}\right)}{3!}\left(\frac{x}{2}\right)^3+\cdots\right]$$

$$= \frac{x^2}{\sqrt{2}} + \frac{x^2}{\sqrt{2}}\sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!\,2^{2n}}x^n$$

$$= \frac{x^2}{\sqrt{2}} + \sum_{n=1}^{\infty}(-1)^n\frac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{n!\,2^{2n+1/2}}x^{n+2} \text{ and } \left|\frac{x}{2}\right| < 1 \quad\Leftrightarrow\quad |x| < 2, \quad \text{so } R = 2.$$

40. $3^\circ = \dfrac{\pi}{60}$ radians and $\sin x = \displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\,x^{2n+1}}{(2n+1)!}$, so

$\sin\dfrac{\pi}{60} = \dfrac{\pi}{60} - \dfrac{\left(\frac{\pi}{60}\right)^3}{3!} + \dfrac{\left(\frac{\pi}{60}\right)^5}{5!} - \cdots = \dfrac{\pi}{60} - \dfrac{\pi^3}{1{,}296{,}000} + \dfrac{\pi^5}{93{,}312{,}000{,}000} - \cdots$. But $\dfrac{\pi^5}{93{,}312{,}000{,}000} < 10^{-8}$, so by the Alternating Series Estimation Theorem, $\sin\dfrac{\pi}{60} \approx \dfrac{\pi}{60} - \dfrac{\pi^3}{1{,}296{,}000} \approx 0.05234$.

44. $\dfrac{\sin x}{x} = \dfrac{1}{x}\displaystyle\sum_{n=0}^{\infty}\frac{(-1)^n\,x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty}\frac{(-1)^n\,x^{2n}}{(2n+1)!}$, so $\displaystyle\int\frac{\sin x}{x}\,dx = \int\sum_{n=0}^{\infty}\frac{(-1)^n\,x^{2n}}{(2n+1)!}\,dx = C + \sum_{n=0}^{\infty}\frac{(-1)^n\,x^{2n+1}}{(2n+1)(2n+1)!}$