4. $-\frac{1}{3}+\frac{2}{4}-\frac{3}{5}+\frac{4}{6}-\frac{5}{7}+\cdots=\sum_{n=1}^{\infty}(-1)^n\frac{n}{n+2}$. Here $a_n=(-1)^n\frac{n}{n+2}$. Since $\lim_{n\to\infty}a_n\neq 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

6. $\sum_{n=1}^{\infty}a_n=\sum_{n=1}^{\infty}(-1)^n\frac{\sqrt{n}}{1+2\sqrt{n}}=\sum_{n=1}^{\infty}(-1)^n b_n$. Now $\lim_{n\to\infty}b_n=\lim_{n\to\infty}\frac{1}{2+1/\sqrt{n}}=\frac{1}{2}\neq 0$. Since $\lim_{n\to\infty}a_n\neq 0$ (in fact the limit does not exist), the series diverges by the Test for Divergence.

8. $\sum_{n=1}^{\infty}(-1)^{n-1}\left(\frac{\ln n}{n}\right)=0+\sum_{n=2}^{\infty}(-1)^{n-1}\left(\frac{\ln n}{n}\right)$. $b_n=\frac{\ln n}{n}>0$ for $n\geq 2$, and if $f(x)=\frac{\ln x}{x}$, then $f'(x)=\frac{1-\ln x}{x^2}<0$ for $x>e$, so $\{b_n\}$ is eventually decreasing. Also,

$$\lim_{n\to\infty}b_n=\lim_{n\to\infty}\frac{\ln n}{n}=\lim_{x\to\infty}\frac{\ln x}{x}\overset{H}{=}\lim_{x\to\infty}\frac{1/x}{1}=0,$$ so the series converges by the Alternating Series Test.

10. The series $\sum_{n=1}^{\infty}\frac{(-1)^n}{n\,5^n}$ satisfies (i) of the Alternating Series Test because $\frac{1}{(n+1)5^{n+1}}<\frac{1}{n\,5^n}$ and (ii) $\lim_{n\to\infty}\frac{1}{n\,5^n}=0$, so the series is convergent. Now $b_4=\frac{1}{4\cdot 5^4}=0.0004>0.0001$ and $b_5=\frac{1}{5\cdot 5^5}=0.000064<0.0001$, so by the Alternating Series Estimation Theorem, $n=4$. (That is, since the 5th term is less than the desired error, we need to add the first 4 terms to get the sum to the desired accuracy.)

16. $b_6=\frac{1}{3^6\cdot 6!}=\frac{1}{524{,}880}\approx 0.000\,001\,9$, so

$\sum_{n=1}^{\infty}\frac{(-1)^n}{3^n n!}\approx s_5=\sum_{n=1}^{5}\frac{(-1)^n}{3^n n!}=-\frac{1}{3}+\frac{1}{18}-\frac{1}{162}+\frac{1}{1944}-\frac{1}{29{,}160}\approx -0.283\,471$. Adding b_6 to s_5 does not change the fourth decimal place of s_5, so the sum of the series, correct to four decimal places, is -0.2835.

18. If $p>0$, $\frac{1}{(n+1)^p}\leq\frac{1}{n^p}$ [$\{1/n^p\}$ is decreasing] and $\lim_{n\to\infty}\frac{1}{n^p}=0$, so the series converges by the Alternating Series Test. If $p\leq 0$, $\lim_{n\to\infty}\frac{(-1)^{n-1}}{n^p}$ does not exist, so the series diverges by the Test for Divergence. Thus,

$\sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^p}$ converges $\Leftrightarrow$ $p>0$.

20. The series $\sum_{n=1}^{\infty}\frac{n^2}{2^n}$ has positive terms and $\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\left[\frac{(n+1)^2}{2^{n+1}}\cdot\frac{2^n}{n^2}\right]=\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^2\cdot\frac{1}{2}=\frac{1}{2}<1$, so the series is absolutely convergent by the Ratio Test.

22. $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$ diverges by the Limit Comparison Test with the harmonic series: $\lim_{n\to\infty} \frac{n/(n^2+1)}{1/n} = \lim_{n\to\infty} \frac{n^2}{n^2+1} = 1$. But $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$ converges by the Alternating Series Test: $\left\{\frac{n}{n^2+1}\right\}$ has positive terms, is decreasing since $\left(\frac{x}{x^2+1}\right)' = \frac{1-x^2}{(x^2+1)^2} \leq 0$ for $x \geq 1$, and $\lim_{n\to\infty} \frac{n}{n^2+1} = 0$. Thus, $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{n}{n^2+1}$ is conditionally convergent.

24. $\sum_{n=1}^{\infty} (-1)^{n-1} \frac{2^n}{n^4}$ diverges by the Test for Divergence. $\lim_{n\to\infty} \frac{2^n}{n^4} = \infty$, so $\lim_{n\to\infty} (-1)^{n-1} \frac{2^n}{n^4}$ does not exist.

26. $\left|\frac{\sin 4n}{4^n}\right| \leq \frac{1}{4^n}$, so $\sum_{n=1}^{\infty} \left|\frac{\sin 4n}{4^n}\right|$ converges by comparison with the convergent geometric series $\sum_{n=1}^{\infty} \frac{1}{4^n}$ $\left[|r| = \frac{1}{4} < 1\right]$. Thus, $\sum_{n=1}^{\infty} \frac{\sin 4n}{4^n}$ is absolutely convergent.

30. $\lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \frac{1}{\ln n} = 0 < 1$, so the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{(\ln n)^n}$ converges absolutely by the Root Test.

32. Since $\left\{\frac{1}{n \ln n}\right\}$ is decreasing and $\lim_{n\to\infty} \frac{1}{n \ln n} = 0$, the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ converges by the Alternating Series Test. Since $\sum_{n=2}^{\infty} \frac{1}{n \ln n}$ diverges by the Integral Test (Exercise 8.3.15), the series $\sum_{n=2}^{\infty} \frac{(-1)^n}{n \ln n}$ is conditionally convergent.

38. $\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{\frac{2^{n+1}(n+1)!}{5\cdot 8\cdot 11\cdot\cdots\cdot(3n+5)}}{\frac{2^n n!}{5\cdot 8\cdot 11\cdot\cdots\cdot(3n+2)}}\right| = \lim_{n\to\infty} \frac{2(n+1)}{3n+5} = \frac{2}{3} < 1$, so the series converges absolutely by the Ratio Test.

40. We use the Ratio Test:

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left|\frac{[(n+1)!]^2/[k(n+1)]!}{(n!)^2/(kn)!}\right| = \lim_{n\to\infty} \left|\frac{(n+1)^2}{[k(n+1)]\,[k(n+1)-1]\cdots[kn+1]}\right|$$

Now if $k = 1$, then this is equal to $\lim_{n\to\infty} \left|\frac{(n+1)^2}{(n+1)}\right| = \infty$, so the series diverges; if $k = 2$, the limit is $\lim_{n\to\infty} \left|\frac{(n+1)^2}{(2n+2)(2n+1)}\right| = \frac{1}{4} < 1$, so the series converges, and if $k > 2$, then the highest power of n in the denominator is larger than 2, and so the limit is 0, indicating convergence. So the series converges for $k \geq 2$.