4. (a) If $a_n > b_n$ for all n, then $\sum a_n$ is divergent. [This is part (ii) of the Comparison Test.]

(b) We cannot say anything about $\sum a_n$. If $a_n < b_n$ for all n and $\sum b_n$ is divergent, then $\sum a_n$ could be convergent or divergent.

6. The function $f(x) = 1/\sqrt[4]{x} = x^{-1/4}$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$\int_1^\infty x^{-1/4}\,dx = \lim_{t\to\infty} \int_1^t x^{-1/4}\,dx = \lim_{t\to\infty} \left[\frac{4}{3}x^{3/4}\right]_1^t = \lim_{t\to\infty} \left(\frac{4}{3}t^{3/4} - \frac{4}{3}\right) = \infty$, so $\sum_{n=1}^\infty 1/\sqrt[4]{n}$ diverges.

8. The function $f(x) = 1/(x^2+1)$ is continuous, positive, and decreasing on $[1, \infty)$, so the Integral Test applies.

$\displaystyle\int_1^\infty \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \left[\tan^{-1} x\right]_1^t = \lim_{t\to\infty} \left(\tan^{-1} t - \tan^{-1} 1\right) = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$, so $\displaystyle\sum_{n=1}^\infty \frac{1}{n^2+1}$ converges.

10. $\displaystyle\frac{\sqrt{n}}{n-1} > \frac{\sqrt{n}}{n} = \frac{1}{\sqrt{n}}$, so $\displaystyle\sum_{n=2}^\infty \frac{\sqrt{n}}{n-1}$ diverges by comparison with the divergent (partial) p-series $\displaystyle\sum_{n=2}^\infty \frac{1}{\sqrt{n}}$ $\left[p = \frac{1}{2} \le 1\right]$.

12. $\displaystyle\sum_{n=1}^\infty \frac{1}{n^4}$ and $\displaystyle\sum_{n=1}^\infty \frac{1}{n^{3/2}}$ are convergent p-series with $p = 4 > 1$ and $p = \frac{3}{2} > 1$, respectively. Thus,

$\displaystyle\sum_{n=1}^\infty \left(\frac{5}{n^4} + \frac{4}{n\sqrt{n}}\right) = 5\sum_{n=1}^\infty \frac{1}{n^4} + 4\sum_{n=1}^\infty \frac{1}{n^{3/2}}$ is convergent by Theorems 8.2.8(i) and 8.2.8(ii).

14. $\displaystyle f(x) = \frac{x^2}{x^3+1}$ is continuous and positive on $[2, \infty)$, and also decreasing since $\displaystyle f'(x) = \frac{x(2-x^3)}{(x^3+1)^2} < 0$ for $x \ge 2$, so we can use the Integral Test [note that f is *not* decreasing on $[1, \infty)$].

$\displaystyle\int_2^\infty \frac{x^2}{x^3+1}\,dx = \lim_{t\to\infty} \left[\frac{1}{3}\ln(x^3+1)\right]_2^t = \frac{1}{3}\lim_{t\to\infty} \left[\ln(t^3+1) - \ln 9\right] = \infty$, so the series $\displaystyle\sum_{n=2}^\infty \frac{n^2}{n^3+1}$ diverges, and so does the given series, $\displaystyle\sum_{n=1}^\infty \frac{n^2}{n^3+1}$.

Another solution: Use the Limit Comparison Test with $\displaystyle a_n = \frac{n^2}{n^3+1}$ and $\displaystyle b_n = \frac{1}{n}$:

$\displaystyle\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n^2 \cdot n}{n^3+1} = \lim_{n\to\infty} \frac{1}{1+1/n^3} = 1 > 0$. Since the harmonic series $\displaystyle\sum_{n=1}^\infty \frac{1}{n}$ diverges, so does $\displaystyle\sum_{n=1}^\infty \frac{n^2}{n^3+1}$.

16. $\displaystyle\frac{n^2-1}{3n^4+1} < \frac{n^2}{3n^4+1} < \frac{n^2}{3n^4} = \frac{1}{3}\frac{1}{n^2}$. $\displaystyle\sum_{n=1}^\infty \frac{n^2-1}{3n^4+1}$ converges by comparison with $\displaystyle\sum_{n=1}^\infty \frac{1}{3n^2}$, which converges because it is a constant multiple of a convergent p-series $[p = 2 > 1]$. The terms of the given series are positive for $n > 1$, which is good enough.

18. $\displaystyle\frac{4+3^n}{2^n} > \frac{3^n}{2^n} = \left(\frac{3}{2}\right)^n$ for all $n \ge 1$, so $\displaystyle\sum_{n=1}^\infty \frac{4+3^n}{2^n}$ diverges by comparison with the divergent geometric series $\displaystyle\sum_{n=1}^\infty \left(\frac{3}{2}\right)^n$.

20. $\dfrac{1}{\sqrt{n^3+1}} < \dfrac{1}{\sqrt{n^3}} = \dfrac{1}{n^{3/2}}$, so $\sum\limits_{n=1}^{\infty} \dfrac{1}{\sqrt{n^3+1}}$ converges by comparison with the convergent p-series $\sum\limits_{n=1}^{\infty} \dfrac{1}{n^{3/2}}$ $\left[p = \frac{3}{2} > 1\right]$.

24. $\dfrac{1+\sin n}{10^n} \le \dfrac{2}{10^n}$ and $\sum\limits_{n=0}^{\infty} \dfrac{2}{10^n} = 2\sum\limits_{n=0}^{\infty}\left(\dfrac{1}{10}\right)^n$, so the given series converges by comparison with a constant multiple of a convergent geometric series.

28. If $p \le 0$, $\lim\limits_{n\to\infty} \dfrac{\ln n}{n^p} = \infty$ and the series diverges, so assume $p > 0$. $f(x) = \dfrac{\ln x}{x^p}$ is positive and continuous and $f'(x) < 0$ for $x > e^{1/p}$, so f is eventually decreasing and we can use the Integral Test. Integration by parts gives

$$\int_1^\infty \frac{\ln x}{x^p}\,dx = \lim_{t\to\infty}\left[\frac{x^{1-p}\left[(1-p)\ln x - 1\right]}{(1-p)^2}\right]_1^t \quad [\text{for } p \ne 1] = \frac{1}{(1-p)^2}\left[\lim_{t\to\infty} t^{1-p}\left[(1-p)\ln t - 1\right] + 1\right]$$

which exists whenever $1 - p < 0 \quad \Leftrightarrow \quad p > 1$. Since we have already done the case $p = 1$ in Exercise 27 (set $p = -1$ in that exercise), $\sum\limits_{n=1}^{\infty} \dfrac{\ln n}{n^p}$ converges $\quad \Leftrightarrow \quad p > 1$.

32. $f(x) = 1/x^5$ is positive and continuous and $f'(x) = -5/x^6$ is negative for $x > 0$, and so the Integral Test applies. Using Exercise 29(a), $R_n \le \int_n^\infty x^{-5}\,dx = \lim\limits_{t\to\infty}\left[\dfrac{-1}{4x^4}\right]_n^t = \dfrac{1}{4n^4}$. If we take $n = 5$, then $s_5 \approx 1.036662$ and $R_5 \le 0.0004$. So $s \approx s_5 \approx 1.037$.

40. Let $a_n = \dfrac{1}{n^2}$ and $b_n = \dfrac{1}{n}$. Then $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = \lim\limits_{n\to\infty} \dfrac{1}{n} = 0$, but $\sum b_n$ diverges while $\sum a_n$ converges.