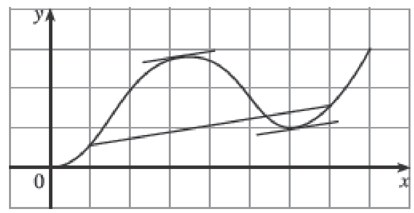
**2.** $f(x) = x^3 - x^2 - 6x + 2$, $[0, 3]$. Since $f$ is a polynomial, it is continuous and differentiable on $\mathbb{R}$, so it is continuous on $[0, 3]$ and differentiable on $(0, 3)$. Also, $f(0) = 2 = f(3)$. $f'(c) = 0 \quad \Leftrightarrow \quad 3c^2 - 2c - 6 = 0 \quad \Leftrightarrow$

$c = \dfrac{2 \pm \sqrt{4 + 72}}{6} = \frac{1}{3} \pm \frac{1}{3}\sqrt{19}$ $[\approx 1.79, \approx -1.12]$, so $c = \frac{1}{3} + \frac{1}{3}\sqrt{19}$ satisfies the conclusion of Rolle's Theorem.

**8.** $f'(c) = \dfrac{f(7) - f(1)}{7 - 1} = \dfrac{1.6 - 0.6}{6} = \dfrac{1}{6}$. It appears that $f'(c) = \frac{1}{6}$ when $c \approx 3.2$ and $6.1$.




**10.** $f(x) = x^3 - 3x + 2$, $[-2, 2]$. $f$ is continuous on $[-2, 2]$ and differentiable on $(-2, 2)$ since polynomials are continuous and differentiable on $\mathbb{R}$. $f'(c) = \dfrac{f(b) - f(a)}{b - a} \quad \Leftrightarrow \quad 3c^2 - 3 = \dfrac{f(2) - f(-2)}{2 - (-2)} = \dfrac{4 - 0}{4} = 1 \quad \Leftrightarrow \quad 3c^2 = 4 \quad \Leftrightarrow$

$c^2 = \dfrac{4}{3} \quad \Leftrightarrow \quad c = \pm\dfrac{2}{\sqrt{3}}$, which are both in $(-2, 2)$.

**12.** $f(x) = \dfrac{1}{x}$, $[1, 3]$. $f$ is continuous and differentiable on $(-\infty, 0) \cup (0, \infty)$, so $f$ is continous on $[1, 3]$ and differentiable on $(1, 3)$. $f'(c) = \dfrac{f(b) - f(a)}{b - a} \quad \Leftrightarrow \quad -\dfrac{1}{c^2} = \dfrac{f(3) - f(1)}{3 - 1} = \dfrac{\frac{1}{3} - 1}{2} = -\dfrac{1}{3} \quad \Leftrightarrow \quad c^2 = 3 \quad \Leftrightarrow \quad c = \pm\sqrt{3}$, but only $\sqrt{3}$ is in $(1, 3)$.

**16.** $f(x) = 2 - |2x - 1| = \begin{cases} 2 - (2x - 1) & \text{if } 2x - 1 \geq 0 \\ 2 - [-(2x - 1)] & \text{if } 2x - 1 < 0 \end{cases} = \begin{cases} 3 - 2x & \text{if } x \geq \frac{1}{2} \\ 1 + 2x & \text{if } x < \frac{1}{2} \end{cases} \quad \Rightarrow \quad f'(x) = \begin{cases} -2 & \text{if } x > \frac{1}{2} \\ 2 & \text{if } x < \frac{1}{2} \end{cases}$

$f(3) - f(0) = f'(c)(3 - 0) \quad \Rightarrow \quad -3 - 1 = f'(c) \cdot 3 \quad \Rightarrow \quad f'(c) = -\frac{4}{3}$ [not $\pm 2$]. This does not contradict the Mean Value Theorem since $f$ is not differentiable at $x = \frac{1}{2}$.

**20.** $f(x) = x^4 + 4x + c$. Suppose that $f(x) = 0$ has three distinct real roots $a$, $b$, $d$ where $a < b < d$. Then $f(a) = f(b) = f(d) = 0$. By Rolle's Theorem there are numbers $c_1$ and $c_2$ with $a < c_1 < b$ and $b < c_2 < d$ and $0 = f'(c_1) = f'(c_2)$, so $f'(x) = 0$ must have at least two real solutions. However $0 = f'(x) = 4x^3 + 4 = 4(x^3 + 1) = 4(x + 1)((x^2 - x + 1)$ has as its only real solution $x = -1$. Thus, $f(x)$ can have at most two real roots.

**24.** If $3 \leq f'(x) \leq 5$ for all $x$, then by the Mean Value Theorem, $f(8) - f(2) = f'(c) \cdot (8 - 2)$ for some $c$ in $[2, 8]$. ($f$ is differentiable for all $x$, so, in particular, $f$ is differentiable on $(2, 8)$ and continuous on $[2, 8]$. Thus, the hypotheses of the Mean Value Theorem are satisfied.) Since $f(8) - f(2) = 6f'(c)$ and $3 \leq f'(c) \leq 5$, it follows that

$6 \cdot 3 \leq 6f'(c) \leq 6 \cdot 5 \quad \Rightarrow \quad 18 \leq f(8) - f(2) \leq 30.$

**28.** $f$ satisfies the conditions for the Mean Value Theorem, so we use this theorem on the interval $[-b, b]$: $\dfrac{f(b) - f(-b)}{b - (-b)} = f'(c)$ for some $c \in (-b, b)$. But since $f$ is odd, $f(-b) = -f(b)$. Substituting this into the above equation, we get

$$\frac{f(b) + f(b)}{2b} = f'(c) \quad \Rightarrow \quad \frac{f(b)}{b} = f'(c).$$

**34.** Let $v(t)$ be the velocity of the car $t$ hours after 2:00 PM. Then $\dfrac{v(1/6) - v(0)}{1/6 - 0} = \dfrac{50 - 30}{1/6} = 120$. By the Mean Value Theorem, there is a number $c$ such that $0 < c < \frac{1}{6}$ with $v'(c) = 120$. Since $v'(t)$ is the acceleration at time $t$, the acceleration $c$ hours after 2:00 PM is exactly $120\ \mathrm{mi/h^2}$.

**2.** (a) $f$ is increasing on $(0, 1)$ and $(3, 7)$.

(b) $f$ is decreasing on $(1, 3)$.

(c) $f$ is concave upward on $(2, 4)$ and $(5, 7)$.

(d) $f$ is concave downward on $(0, 2)$ and $(4, 5)$.

(e) The points of inflection are $(2, 2)$, $(4, 3)$, and $(5, 4)$.

**6.** (a) $f'(x) > 0$ and $f$ is increasing on $(0, 1)$ and $(3, 5)$. $f'(x) < 0$ and $f$ is decreasing on $(1, 3)$ and $(5, 6)$.

(b) Since $f'(x) = 0$ at $x = 1$ and $x = 5$ and $f'$ changes from positive to negative at both values, $f$ changes from increasing to decreasing and has local maxima at $x = 1$ and $x = 5$. Since $f'(x) = 0$ at $x = 3$ and $f'$ changes from negative to positive there, $f$ changes from decreasing to increasing and has a local minimum at $x = 3$.

**10.** (a) $f(x) = 4x^3 + 3x^2 - 6x + 1 \quad \Rightarrow \quad f'(x) = 12x^2 + 6x - 6 = 6(2x^2 + x - 1) = 6(2x - 1)(x + 1)$. Thus, $f'(x) > 0 \quad \Leftrightarrow \quad x < -1$ or $x > \frac{1}{2}$ and $f'(x) < 0 \quad \Leftrightarrow \quad -1 < x < \frac{1}{2}$. So $f$ is increasing on $(-\infty, -1)$ and $\left(\frac{1}{2}, \infty\right)$ and $f$ is decreasing on $\left(-1, \frac{1}{2}\right)$.

(b) $f$ changes from increasing to decreasing at $x = -1$ and from decreasing to increasing at $x = \frac{1}{2}$. Thus, $f(-1) = 6$ is a local maximum value and $f\left(\frac{1}{2}\right) = -\frac{3}{4}$ is a local minimum value.

(c) $f''(x) = 24x + 6 = 6(4x + 1)$. $\quad f''(x) > 0 \quad \Leftrightarrow \quad x > -\frac{1}{4}$ and $f''(x) < 0 \quad \Leftrightarrow \quad x < -\frac{1}{4}$. Thus, $f$ is concave upward on $\left(-\frac{1}{4}, \infty\right)$ and concave downward on $\left(-\infty, -\frac{1}{4}\right)$. There is an inflection point at $\left(-\frac{1}{4}, f\left(-\frac{1}{4}\right)\right) = \left(-\frac{1}{4}, \frac{21}{8}\right)$.

**14.** (a) $f(x) = \cos^2 x - 2\sin x,\ \ 0 \le x \le 2\pi.\quad f'(x) = -2\cos x\,\sin x - 2\cos x = -2\cos x\,(1 + \sin x)$. Note that $1 + \sin x \ge 0$ [since $\sin x \ge -1$], with equality $\Leftrightarrow \sin x = -1 \Leftrightarrow x = \frac{3\pi}{2}$ [since $0 \le x \le 2\pi$] $\Rightarrow \cos x = 0$. Thus, $f'(x) > 0 \Leftrightarrow \cos x < 0 \Leftrightarrow \frac{\pi}{2} < x < \frac{3\pi}{2}$ and $f'(x) < 0 \Leftrightarrow \cos x > 0 \Leftrightarrow 0 < x < \frac{\pi}{2}$ or $\frac{3\pi}{2} < x < 2\pi$. Thus, $f$ is increasing on $\left(\frac{\pi}{2}, \frac{3\pi}{2}\right)$ and $f$ is decreasing on $\left(0, \frac{\pi}{2}\right)$ and $\left(\frac{3\pi}{2}, 2\pi\right)$.

(b) $f$ changes from decreasing to increasing at $x = \frac{\pi}{2}$ and from increasing to decreasing at $x = \frac{3\pi}{2}$. Thus, $f\left(\frac{\pi}{2}\right) = -2$ is a local minimum value and $f\left(\frac{3\pi}{2}\right) = 2$ is a local maximum value.

(c)
$$\begin{aligned} f''(x) &= 2\sin x\,(1 + \sin x) - 2\cos^2 x = 2\sin x + 2\sin^2 x - 2(1 - \sin^2 x) \\ &= 4\sin^2 x + 2\sin x - 2 = 2(2\sin x - 1)(\sin x + 1) \end{aligned}$$

so $f''(x) > 0 \Leftrightarrow \sin x > \frac{1}{2} \Leftrightarrow \frac{\pi}{6} < x < \frac{5\pi}{6}$, and $f''(x) < 0 \Leftrightarrow \sin x < \frac{1}{2}$ and $\sin x \ne -1 \Leftrightarrow 0 < x < \frac{\pi}{6}$ or $\frac{5\pi}{6} < x < \frac{3\pi}{2}$ or $\frac{3\pi}{2} < x < 2\pi$. Thus, $f$ is concave upward on $\left(\frac{\pi}{6}, \frac{5\pi}{6}\right)$ and concave downward on $\left(0, \frac{\pi}{6}\right)$, $\left(\frac{5\pi}{6}, \frac{3\pi}{2}\right)$, and $\left(\frac{3\pi}{2}, 2\pi\right)$. There are inflection points at $\left(\frac{\pi}{6}, -\frac{1}{4}\right)$ and $\left(\frac{5\pi}{6}, -\frac{1}{4}\right)$.

**16.** (a) $f(x) = x^2 \ln x \Rightarrow f'(x) = x^2(1/x) + (\ln x)(2x) = x + 2x\ln x = x(1 + 2\ln x)$. The domain of $f$ is $(0, \infty)$, so the sign of $f'$ is determined solely by the factor $1 + 2\ln x$. $f'(x) > 0 \Leftrightarrow \ln x > -\frac{1}{2} \Leftrightarrow x > e^{-1/2}$ [$\approx 0.61$] and $f'(x) < 0 \Leftrightarrow 0 < x < e^{-1/2}$. So $f$ is increasing on $(e^{-1/2}, \infty)$ and $f$ is decreasing on $(0, e^{-1/2})$.

(b) $f$ changes from decreasing to increasing at $x = e^{-1/2}$. Thus, $f(e^{-1/2}) = (e^{-1/2})^2 \ln(e^{-1/2}) = e^{-1}(-1/2) = -1/(2e)$ [$\approx -0.18$] is a local minimum value.

(c) $f'(x) = x(1 + 2\ln x) \Rightarrow f''(x) = x(2/x) + (1 + 2\ln x)\cdot 1 = 2 + 1 + 2\ln x = 3 + 2\ln x$. $f''(x) > 0 \Leftrightarrow 3 + 2\ln x > 0 \Leftrightarrow \ln x > -3/2 \Leftrightarrow x > e^{-3/2}$ [$\approx 0.22$]. Thus, $f$ is concave upward on $(e^{-3/2}, \infty)$ and $f$ is concave downward on $(0, e^{-3/2})$. $f(e^{-3/2}) = (e^{-3/2})^2 \ln e^{-3/2} = e^{-3}(-3/2) = -3/(2e^3)$ [$\approx -0.07$]. There is a point of inflection at $\left(e^{-3/2}, f(e^{-3/2})\right) = \left(e^{-3/2}, -3/(2e^3)\right)$.

**20.** $f(x) = \dfrac{x^2}{x-1} \quad \Rightarrow \quad f'(x) = \dfrac{(x-1)(2x) - x^2(1)}{(x-1)^2} = \dfrac{x^2 - 2x}{(x-1)^2} = \dfrac{x(x-2)}{(x-1)^2}$.

*First Derivative Test:* $f'(x) > 0 \quad \Rightarrow \quad x < 0$ or $x > 2$ and $f'(x) < 0 \quad \Rightarrow \quad 0 < x < 1$ or $1 < x < 2$. Since $f'$ changes from positive to negative at $x = 0$, $f(0) = 0$ is a local maximum value; and since $f'$ changes from negative to positive at $x = 2$, $f(2) = 4$ is a local minimum value.

*Second Derivative Test:*

$$f''(x) = \frac{(x-1)^2(2x-2) - (x^2-2x)2(x-1)}{[(x-1)^2]^2} = \frac{2(x-1)[(x-1)^2 - (x^2-2x)]}{(x-1)^4} = \frac{2}{(x-1)^3}.$$

$f'(x) = 0 \quad \Leftrightarrow \quad x = 0, 2$. $f''(0) = -2 < 0 \quad \Rightarrow \quad f(0) = 0$ is a local maximum value. $f''(2) = 2 > 0 \quad \Rightarrow \quad f(2) = 4$ is a local minimum value.

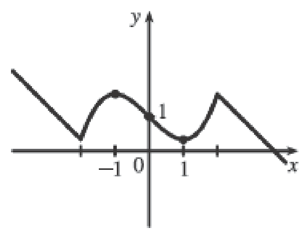
*Preference:* Since calculating the second derivative is fairly difficult, the First Derivative Test is easier to use for this function.

**26.** $f'(1) = f'(-1) = 0 \quad \Rightarrow \quad$ horizontal tangents at $x = \pm 1$.

$f'(x) < 0$ if $|x| < 1 \quad \Rightarrow \quad f$ is decreasing on $(-1, 1)$.

$f'(x) > 0$ if $1 < |x| < 2 \quad \Rightarrow \quad f$ is increasing on $(-2, -1)$ and $(1, 2)$.

$f'(x) = -1$ if $|x| > 2 \quad \Rightarrow \quad$ the graph of $f$ has constant slope $-1$ on $(-\infty, -2)$ and $(2, \infty)$.




$f''(x) < 0$ if $-2 < x < 0 \quad \Rightarrow \quad f$ is concave downward on $(-2, 0)$. The point $(0, 1)$ is an inflection point.

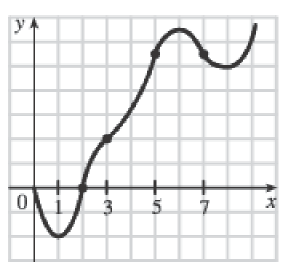
**32.** (a) $f$ is increasing where $f'$ is positive, on $(1, 6)$ and $(8, \infty)$, and decreasing where $f'$ is negative, on $(0, 1)$ and $(6, 8)$.

(b) $f$ has a local maximum where $f'$ changes from positive to negative, at $x = 6$, and local minima where $f'$ changes from negative to positive, at $x = 1$ and at $x = 8$.

(c) $f$ is concave upward where $f'$ is increasing, that is, on $(0, 2)$, $(3, 5)$, and $(7, \infty)$, and concave downward where $f'$ is decreasing, that is, on $(2, 3)$ and $(5, 7)$.

(d) There are points of inflection where $f$ changes its direction of concavity, at $x = 2$, $x = 3$, $x = 5$ and $x = 7$.

(e)

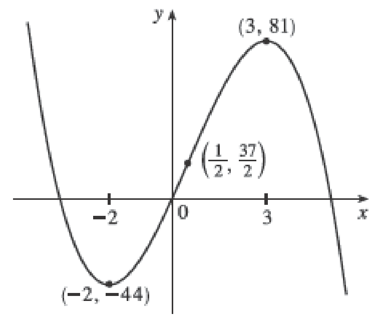


**34.** (a) $f(x) = 36x + 3x^2 - 2x^3 \quad \Rightarrow \quad f'(x) = 36 + 6x - 6x^2 = -6(x^2 - x - 6) = -6(x+2)(x-3)$. $f'(x) > 0 \quad \Leftrightarrow$ $-2 < x < 3$ and $f'(x) < 0 \quad \Leftrightarrow \quad x < -2$ or $x > 3$. So $f$ is increasing on $(-2, 3)$ and $f$ is decreasing on $(-\infty, -2)$ and $(3, \infty)$.

(b) $f$ changes from increasing to decreasing at $x = 3$, so $f(3) = 81$ is a local maximum value. $f$ changes from decreasing to increasing at $x = -2$, so $f(-2) = -44$ is a local minimum value.

(c) $f''(x) = 6 - 12x$. $f''(x) = 0 \quad \Leftrightarrow \quad x = \frac{1}{2}$. $f''(x) > 0$ on $\left(-\infty, \frac{1}{2}\right)$ and $f''(x) < 0$ on $\left(\frac{1}{2}, \infty\right)$. So $f$ is CU on $\left(-\infty, \frac{1}{2}\right)$ and $f$ is CD on $\left(\frac{1}{2}, \infty\right)$. There is an inflection point at $\left(\frac{1}{2}, \frac{37}{2}\right)$.

(d)

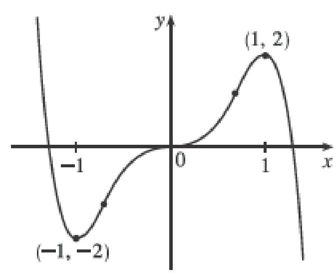



**38.** (a) $h(x) = 5x^3 - 3x^5 \quad \Rightarrow \quad h'(x) = 15x^2 - 15x^4 = 15x^2(1 - x^2) = 15x^2(1+x)(1-x)$. $h'(x) > 0 \quad \Leftrightarrow$ $-1 < x < 0$ and $0 < x < 1$ [note that $h'(0) = 0$] and $h'(x) < 0 \quad \Leftrightarrow \quad x < -1$ or $x > 1$. So $h$ is increasing on $(-1, 1)$ and $h$ is decreasing on $(-\infty, -1)$ and $(1, \infty)$.

(b) $h$ changes from decreasing to increasing at $x = -1$, so $h(-1) = -2$ is a local minimum value. $h$ changes from increasing to decreasing at $x = 1$, so $h(1) = 2$ is a local maximum value.

(c) $h''(x) = 30x - 60x^3 = 30x(1 - 2x^2)$. $h''(x) = 0 \quad \Leftrightarrow \quad x = 0$ or $1 - 2x^2 = 0 \quad \Leftrightarrow \quad x = 0$ or $x = \pm 1/\sqrt{2}$. $h''(x) > 0$ on $\left(-\infty, -1/\sqrt{2}\right)$ and $\left(0, 1/\sqrt{2}\right)$, and $h''(x) < 0$ on $\left(-1/\sqrt{2}, 0\right)$ and $\left(1/\sqrt{2}, \infty\right)$. So $h$ is CU on $\left(-\infty, -1/\sqrt{2}\right)$ and $\left(0, 1/\sqrt{2}\right)$, and $h$ is CD on $\left(-1/\sqrt{2}, 0\right)$ and $\left(1/\sqrt{2}, \infty\right)$. There are inflection points at $\left(-1/\sqrt{2}, -7/\left(4\sqrt{2}\right)\right)$, $(0, 0)$, and $\left(1/\sqrt{2}, 7/\left(4\sqrt{2}\right)\right)$.

(d)



**48.** $f(x) = \dfrac{e^x}{1-e^x}$ has domain $\{x \mid 1-e^x \neq 0\} = \{x \mid e^x \neq 1\} = \{x \mid x \neq 0\}$.

(a) $\lim\limits_{x\to\infty} \dfrac{e^x}{1-e^x} = \lim\limits_{x\to\infty} \dfrac{e^x/e^x}{(1-e^x)/e^x} = \lim\limits_{x\to\infty} \dfrac{1}{1/e^x - 1} = \dfrac{1}{0-1} = -1$, so $y=-1$ is a HA.

$\lim\limits_{x\to-\infty} \dfrac{e^x}{1-e^x} = \dfrac{0}{1-0} = 0$, so $y=0$ is a HA. $\lim\limits_{x\to 0^+} \dfrac{e^x}{1-e^x} = -\infty$ and $\lim\limits_{x\to 0^-} \dfrac{e^x}{1-e^x} = \infty$, so $x=0$ is a VA.

(b) $f'(x) = \dfrac{(1-e^x)e^x - e^x(-e^x)}{(1-e^x)^2} = \dfrac{e^x[(1-e^x)+e^x]}{(1-e^x)^2} = \dfrac{e^x}{(1-e^x)^2}$. $f'(x) > 0$ for $x \neq 0$, so $f$ is increasing on $(-\infty, 0)$ and $(0, \infty)$.

(c) There is no local maximum or minimum.

(d) $f''(x) = \dfrac{(1-e^x)^2 e^x - e^x \cdot 2(1-e^x)(-e^x)}{[(1-e^x)^2]^2}$

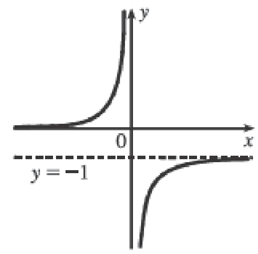
$= \dfrac{(1-e^x)e^x[(1-e^x)+2e^x]}{(1-e^x)^4} = \dfrac{e^x(e^x+1)}{(1-e^x)^3}$

$f''(x) > 0 \iff (1-e^x)^3 > 0 \iff e^x < 1 \iff x < 0$ and

$f''(x) < 0 \iff x > 0$. So $f$ is CU on $(-\infty, 0)$ and $f$ is CD on $(0, \infty)$.

There is no inflection point.

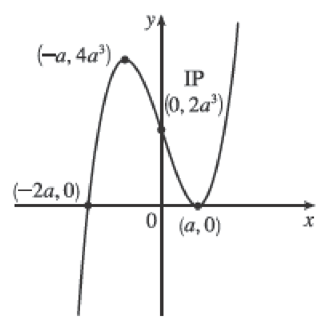
(e)




**54.** $y = f(x) = x^3 - 3a^2x + 2a^3$, $a > 0$. The $y$-intercept is $f(0) = 2a^3$. $y' = 3x^2 - 3a^2 = 3(x^2 - a^2) = 3(x+a)(x-a)$. The critical numbers are $-a$ and $a$. $f' < 0$ on $(-a, a)$, so $f$ is decreasing on $(-a, a)$ and $f$ is increasing on $(-\infty, -a)$ and $(a, \infty)$. $f(-a) = 4a^3$ is a local maximum value and $f(a) = 0$ is a local minimum value. Since $f(a) = 0$, $a$ is an $x$-intercept, and $x - a$ is a factor of $f$. Synthetically dividing $y = x^3 - 3a^2x + 2a^3$ by $x - a$ gives us the following result:

$y = x^3 - 3a^2x + 2a^3 = (x-a)(x^2 + ax - 2a^2) = (x-a)(x-a)(x+2a) = (x-a)^2(x+2a)$, which tells us that the only $x$-intercepts are $-2a$ and $a$. $y' = 3x^2 - 3a^2 \Rightarrow y'' = 6x$, so $y'' > 0$ on $(0, \infty)$ and $y'' < 0$ on $(-\infty, 0)$. This tells us that $f$ is CU on $(0, \infty)$ and CD on $(-\infty, 0)$. There is an inflection point at $(0, 2a^3)$. The graph illustrates these features.

What the curves in the family have in common is that they are all CD on $(-\infty, 0)$, CU on $(0, \infty)$, and have the same basic shape. But as $a$ increases, the four key points shown in the figure move further away from the origin.



**70.** The original equation can be written as $(x^2+b)y+ax=0$. Call this (**1**). Since $(2,2.5)$ is on this curve, we have $(4+b)\left(\frac{5}{2}\right)+2a=0$, or $20+5b+4a=0$. Let's rewrite that as $4a+5b=-20$ and call it (**A**). Differentiating (**1**) gives (after regrouping) $(x^2+b)y'=-(2xy+a)$. Call this (**2**). Differentiating again gives $(x^2+b)y''+(2x)y'=-2xy'-2y$, or $(x^2+b)y''+4xy'+2y=0$. Call this (**3**). At $(2,2.5)$, equations (**2**) and (**3**) say that $(4+b)y'=-(10+a)$ and $(4+b)y''+8y'+5=0$. If $(2,2.5)$ is an inflection point, then $y''=0$ there, so the second condition becomes $8y'+5=0$, or $y'=-\frac{5}{8}$. Substituting in the first condition, we get $-(4+b)\frac{5}{8}=-(10+a)$, or $20+5b=80+8a$, which simplifies to $-8a+5b=60$. Call this (**B**). Subtracting (**B**) from (**A**) yields $12a=-80$, so $a=-\frac{20}{3}$. Substituting that value in (**A**) gives $-\frac{80}{3}+5b=-20=-\frac{60}{3}$, so $5b=\frac{20}{3}$ and $b=\frac{4}{3}$. Thus far we've shown that IF the curve has an inflection point at $(2,2.5)$, then $a=-\frac{20}{3}$ and $b=\frac{4}{3}$.

To prove the converse, suppose that $a=-\frac{20}{3}$ and $b=\frac{4}{3}$. Then by (**1**), (**2**), and (**3**), our curve satisfies

$$\left(x^2+\tfrac{4}{3}\right)y=\tfrac{20}{3}x \qquad (4)$$

$$\left(x^2+\tfrac{4}{3}\right)y'=-2xy+\tfrac{20}{3} \qquad (5)$$

and

$$\left(x^2+\tfrac{4}{3}\right)y''+4xy'+2y=0. \qquad (6)$$

Multiply (**6**) by $\left(x^2+\frac{4}{3}\right)$ and substitute from (**4**) and (**5**) to obtain $\left(x^2+\frac{4}{3}\right)^2y''+4x\left(-2xy+\frac{20}{3}\right)+2\left(\frac{20}{3}x\right)=0$, or $\left(x^2+\frac{4}{3}\right)^2y''-8x^2y+40x=0$. Now multiply by $(x^2+b)$ again and substitute from the first equation to obtain $\left(x^2+\frac{4}{3}\right)^3y''-8x^2\left(\frac{20}{3}x\right)+40x\left(x^2+\frac{4}{3}\right)=0$, or $\left(x^2+\frac{4}{3}\right)^3y''-\frac{40}{3}(x^3-4x)=0$. The coefficient of $y''$ is positive, so the sign of $y''$ is the same as the sign of $\frac{40}{3}(x^3-4x)$, which is a positive multiple of $x(x+2)(x-2)$. It is clear from this expression that $y''$ changes sign at $x=0$, $x=-2$, and $x=2$, so the curve changes its direction of concavity at those values of $x$. By (**4**), the corresponding $y$-values are 0, $-2.5$, and 2.5, respectively. Thus when $a=-\frac{20}{3}$ and $b=\frac{4}{3}$, the curve has inflection points, not only at $(2,2.5)$, but also at $(0,0)$ and $(-2,-2.5)$.

**4.** (a) $\lim\limits_{x\to a}[f(x)]^{g(x)}$ is an indeterminate form of type $0^0$.

(b) If $y=[f(x)]^{p(x)}$, then $\ln y=p(x)\ln f(x)$. When $x$ is near $a$, $p(x)\to\infty$ and $\ln f(x)\to-\infty$, so $\ln y\to-\infty$. Therefore, $\lim\limits_{x\to a}[f(x)]^{p(x)}=\lim\limits_{x\to a}y=\lim\limits_{x\to a}e^{\ln y}=0$, provided $f^p$ is defined.

(c) $\lim\limits_{x\to a}[h(x)]^{p(x)}$ is an indeterminate form of type $1^\infty$.

(d) $\lim\limits_{x\to a}[p(x)]^{f(x)}$ is an indeterminate form of type $\infty^0$.

(e) If $y=[p(x)]^{q(x)}$, then $\ln y=q(x)\ln p(x)$. When $x$ is near $a$, $q(x)\to\infty$ and $\ln p(x)\to\infty$, so $\ln y\to\infty$. Therefore, $\lim\limits_{x\to a}[p(x)]^{q(x)}=\lim\limits_{x\to a}y=\lim\limits_{x\to a}e^{\ln y}=\infty$.

(f) $\lim\limits_{x\to a}\sqrt[q(x)]{p(x)}=\lim\limits_{x\to a}[p(x)]^{1/q(x)}$ is an indeterminate form of type $\infty^0$.

**6.** From the graphs of $f$ and $g$, we see that $\lim_{x\to 2} f(x) = 0$ and $\lim_{x\to 2} g(x) = 0$, so l'Hospital's Rule applies.

$$\lim_{x\to 2}\frac{f(x)}{g(x)} = \lim_{x\to 2}\frac{f'(x)}{g'(x)} = \frac{\lim_{x\to 2} f'(x)}{\lim_{x\to 2} g'(x)} = \frac{f'(2)}{g'(2)} = \frac{1.5}{-1} = -\frac{3}{2}$$

**10.** This limit has the form $\frac{0}{0}$. $\lim_{x\to 1/2}\frac{6x^2+5x-4}{4x^2+16x-9} \overset{\text{H}}{=} \lim_{x\to 1/2}\frac{12x+5}{8x+16} = \frac{6+5}{4+16} = \frac{11}{20}$

*Note:* Alternatively, we could factor and simplify.

**14.** This limit has the form $\frac{0}{0}$. $\lim_{x\to 0}\frac{x^2}{1-\cos x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{2x}{\sin x} = \lim_{x\to 0}\frac{2}{\left(\frac{\sin x}{x}\right)} = \frac{2}{1} = 2$

**20.** This limit has the form $\frac{\infty}{\infty}$. $\lim_{x\to\infty}\frac{\ln\sqrt{x}}{x^2} = \lim_{x\to\infty}\frac{\frac{1}{2}\ln x}{x^2} \overset{\text{H}}{=} \lim_{x\to\infty} = \frac{\frac{1}{2x}}{2x} = \lim_{x\to\infty}\frac{1}{4x^2} = 0$

**26.** This limit has the form $\frac{0}{0}$. $\lim_{x\to 0}\frac{\sinh x - x}{x^3} \overset{\text{H}}{=} \lim_{x\to 0}\frac{\cosh x - 1}{3x^2} \overset{\text{H}}{=} \lim_{x\to 0}\frac{\sinh x}{6x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{\cosh x}{6} = \frac{1}{6}$

**32.** This limit has the form $\frac{0}{0}$.

$$\lim_{x\to 0}\frac{\cos mx - \cos nx}{x^2} \overset{\text{H}}{=} \lim_{x\to 0}\frac{-m\sin mx + n\sin nx}{2x} \overset{\text{H}}{=} \lim_{x\to 0}\frac{-m^2\cos mx + n^2\cos nx}{2} = \tfrac{1}{2}\left(n^2 - m^2\right)$$

**42.** This limit has the form $\infty \cdot 0$. We'll change it to the form $\frac{\infty}{\infty}$.

$$\lim_{x\to\infty}\sqrt{x}\,e^{-x/2} = \lim_{x\to\infty}\frac{\sqrt{x}}{e^{x/2}} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{\frac{1}{2}x^{-1/2}}{\frac{1}{2}e^{x/2}} = \lim_{x\to\infty}\frac{1}{\sqrt{x}\,e^{x/2}} = 0$$

**58.** $y = \left(1+\frac{a}{x}\right)^{bx} \;\Rightarrow\; \ln y = bx\ln\left(1+\frac{a}{x}\right)$, so

$$\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{b\ln(1+a/x)}{1/x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{b\left(\frac{1}{1+a/x}\right)\left(-\frac{a}{x^2}\right)}{-1/x^2} = \lim_{x\to\infty}\frac{ab}{1+a/x} = ab \;\Rightarrow$$

$$\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx} = \lim_{x\to\infty} e^{\ln y} = e^{ab}.$$

**62.** $y = (e^x + x)^{1/x} \;\Rightarrow\; \ln y = \frac{1}{x}\ln(e^x + x)$,

so $\lim_{x\to\infty}\ln y = \lim_{x\to\infty}\frac{\ln(e^x+x)}{x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{e^x+1}{e^x+x} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{e^x}{e^x+1} \overset{\text{H}}{=} \lim_{x\to\infty}\frac{e^x}{e^x} = 1 \;\Rightarrow$

$$\lim_{x\to\infty}(e^x+x)^{1/x} = \lim_{x\to\infty} e^{\ln y} = e^1 = e.$$

**82.** Let the radius of the circle be $r$. We see that $A(\theta)$ is the area of the whole figure (a sector of the circle with radius 1), minus the area of $\triangle OPR$. But the area of the sector of the circle is $\frac{1}{2}r^2\theta$ (see Reference Page 1), and the area of the triangle is $\frac{1}{2}r\,|PQ| = \frac{1}{2}r(r\sin\theta) = \frac{1}{2}r^2\sin\theta$. So we have $A(\theta) = \frac{1}{2}r^2\theta - \frac{1}{2}r^2\sin\theta = \frac{1}{2}r^2(\theta - \sin\theta)$. Now by elementary trigonometry, $B(\theta) = \frac{1}{2}\,|QR|\,|PQ| = \frac{1}{2}(r - |OQ|)\,|PQ| = \frac{1}{2}(r - r\cos\theta)(r\sin\theta) = \frac{1}{2}r^2(1-\cos\theta)\sin\theta$.

So the limit we want is

$$\begin{aligned}
\lim_{\theta\to 0^+}\frac{A(\theta)}{B(\theta)} &= \lim_{\theta\to 0^+}\frac{\frac{1}{2}r^2(\theta-\sin\theta)}{\frac{1}{2}r^2(1-\cos\theta)\sin\theta} \overset{\text{H}}{=} \lim_{\theta\to 0^+}\frac{1-\cos\theta}{(1-\cos\theta)\cos\theta + \sin\theta\,(\sin\theta)} \\
&= \lim_{\theta\to 0^+}\frac{1-\cos\theta}{\cos\theta - \cos^2\theta + \sin^2\theta} \overset{\text{H}}{=} \lim_{\theta\to 0^+}\frac{\sin\theta}{-\sin\theta - 2\cos\theta\,(-\sin\theta) + 2\sin\theta\,(\cos\theta)} \\
&= \lim_{\theta\to 0^+}\frac{\sin\theta}{-\sin\theta + 4\sin\theta\,\cos\theta} = \lim_{\theta\to 0^+}\frac{1}{-1+4\cos\theta} = \frac{1}{-1+4\cos 0} = \frac{1}{3}
\end{aligned}$$