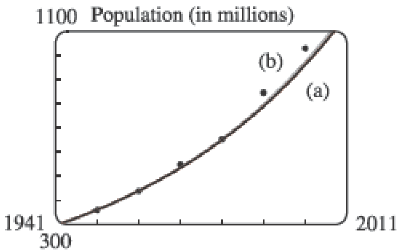
6. (a) Let $P(t)$ be the population (in millions) in the year t. Since the initial time is the year 1951, we substitute $t - 1951$ for t in Theorem 2, and find that the exponential model gives $P(t) = P(1951)e^{k(t-1951)} \Rightarrow$ $P(1961) = 92 = 76e^{k(1961-1951)} \Rightarrow k = \frac{1}{10}\ln\frac{439}{361} \approx 0.0196$. With this model, we estimate $P(2001) = 361e^{k(2001-1951)} \approx 960$ million. This estimate is slightly lower than the given value, 1029 million.

(b) Substituting $t - 1961$ for t in Theorem 2, we find that the exponential model gives $P(t) = P(1961)e^{k(t-1961)} \Rightarrow$ $P(1981) = 653 = 439e^{k(1981-1961)} \Rightarrow k = \frac{1}{20}\ln\frac{653}{439} \approx 0.0199$. With this model, we estimate $P(2001) = 439e^{k(2001-1961)} \approx 971$ million, which is better than the estimate in part (a). The further estimates are $P(2010) = 439e^{49k} \approx 1161$ million and $P(2020) = 439e^{49k} \approx 1416$ million.

(c)



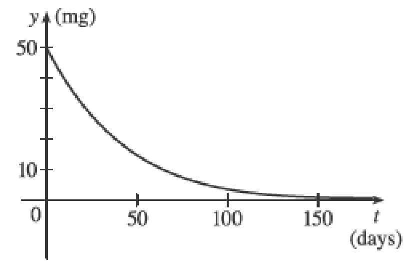
Both models are reasonable; in fact, their graphs are nearly indistinguishable.

8. (a) The mass remaining after t days is $y(t) = y(0)\,e^{kt} = 50e^{kt}$. Since the half-life is 28 days, $y(28) = 50e^{28k} = 25 \Rightarrow$ $e^{28k} = \frac{1}{2} \Rightarrow 28k = \ln\frac{1}{2} \Rightarrow k = -(\ln 2)/28$, so $y(t) = 50e^{-(\ln 2)t/28} = 50 \cdot 2^{-t/28}$.

(b) $y(40) = 50 \cdot 2^{-40/28} \approx 18.6$ mg

(c) $y(t) = 2 \Rightarrow 2 = 50 \cdot 2^{-t/28} \Rightarrow \frac{2}{50} = 2^{-t/28} \Rightarrow$ $(-t/28)\ln 2 = \ln\frac{1}{25} \Rightarrow t = \left(-28\ln\frac{1}{25}\right)/\ln 2 \approx 130$ days

(d)



10. (a) If $y(t)$ is the mass after t days and $y(0) = A$, then $y(t) = Ae^{kt}$.

$y(1) = Ae^{k} = 0.945A \Rightarrow e^{k} = 0.945 \Rightarrow k = \ln 0.945$.

Then $Ae^{(\ln 0.945)t} = \frac{1}{2}A \Leftrightarrow \ln e^{(\ln 0.945)t} = \ln\frac{1}{2} \Leftrightarrow (\ln 0.945)t = \ln\frac{1}{2} \Leftrightarrow t = -\frac{\ln 2}{\ln 0.945} \approx 12.25$ years.

(b) $Ae^{(\ln 0.945)t} = 0.20A \Leftrightarrow (\ln 0.945)t = \ln\frac{1}{5} \Leftrightarrow t = -\frac{\ln 5}{\ln 0.945} \approx 28.45$ years

12. From the information given, we know that $\frac{dy}{dx} = 2y \Rightarrow y = Ce^{2x}$ by Theorem 2. To calculate C we use the point $(0, 5)$: $5 = Ce^{2(0)} \Rightarrow C = 5$. Thus, the equation of the curve is $y = 5e^{2x}$.

16. $\dfrac{dT}{dt} = k(T-20)$. Let $y = T - 20$. Then $\dfrac{dy}{dt} = ky$, so $y(t) = y(0)e^{kt}$. $y(0) = T(0) - 20 = 95 - 20 = 75$,

so $y(t) = 75e^{kt}$. When $T(t) = 70$, $\dfrac{dT}{dt} = -1°\text{C/min}$. Equivalently, $\dfrac{dy}{dt} = -1$ when $y(t) = 50$. Thus,

$-1 = \dfrac{dy}{dt} = ky(t) = 50k$ and $50 = y(t) = 75e^{kt}$. The first relation implies $k = -1/50$, so the second relation says

$50 = 75e^{-t/50}$. Thus, $e^{-t/50} = \frac{2}{3} \Rightarrow -t/50 = \ln\left(\frac{2}{3}\right) \Rightarrow t = -50\ln\left(\frac{2}{3}\right) \approx 20.27$ min.

18. (a) Using $A = A_0\left(1 + \dfrac{r}{n}\right)^{nt}$ with $A_0 = 1000$, $r = 0.08$, and $t = 3$, we have:

(i) Annually: $n = 1$; $A = 1000\left(1 + \frac{0.08}{1}\right)^{1\cdot 3} = \$1259.71$

(ii) Quarterly: $n = 4$; $A = 1000\left(1 + \frac{0.08}{4}\right)^{4\cdot 3} = \$1268.24$

(iii) Monthly: $n = 12$; $A = 1000\left(1 + \frac{0.08}{12}\right)^{12\cdot 3} = \$1270.24$

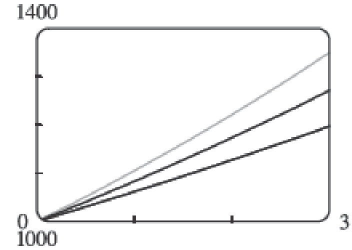
(iv) Weekly: $n = 52$ $A = 1000\left(1 + \frac{0.08}{52}\right)^{52\cdot 3} = \$1271.01$

(v) Daily: $n = 365$; $A = 1000\left(1 + \frac{0.08}{365}\right)^{365\cdot 3} = \$1271.22$

(vi) Hourly: $n = 365 \cdot 24$; $A = 1000\left(1 + \frac{0.08}{365\cdot 24}\right)^{365\cdot 24\cdot 3} = \$1271.25$

(vii) Continuously: $A = 1000e^{(0.08)3} = \$1271.25$

(b)



$A_{0.10}(3) = \$1349.86$,

$A_{0.08}(3) = \$1271.25$, and

$A_{0.06}(3) = \$1197.22$.

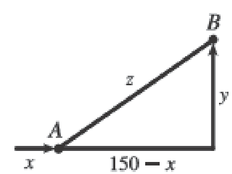
10. $\dfrac{d}{dt}(xy) = \dfrac{d}{dt}(8) \Rightarrow x\dfrac{dy}{dt} + y\dfrac{dx}{dt} = 0$. If $\dfrac{dy}{dt} = -3$ cm/s and $(x, y) = (4, 2)$, then $4(-3) + 2\dfrac{dx}{dt} = 0 \Rightarrow$

$\dfrac{dx}{dt} = 6$. Thus, the x-coordinate is increasing at a rate of 6 cm/s.

14. (a) Given: at noon, ship A is 150 km west of ship B; ship A is sailing east at 35 km/h, and ship B is sailing north at 25 km/h. If we let t be time (in hours), x be the distance traveled by ship A (in km), and y be the distance traveled by ship B (in km), then we are given that $dx/dt = 35$ km/h and $dy/dt = 25$ km/h.

(b) Unknown: the rate at which the distance between the ships is changing at 4:00 PM. If we let z be the distance between the ships, then we want to find dz/dt when $t = 4$ h.

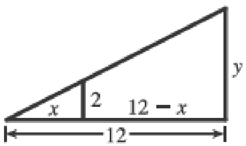
(c)



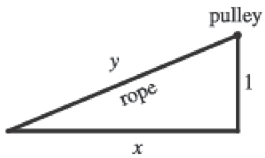
(d) $z^2 = (150 - x)^2 + y^2 \Rightarrow 2z\dfrac{dz}{dt} = 2(150 - x)\left(-\dfrac{dx}{dt}\right) + 2y\dfrac{dy}{dt}$

(e) At 4:00 PM, $x = 4(35) = 140$ and $y = 4(25) = 100 \Rightarrow z = \sqrt{(150 - 140)^2 + 100^2} = \sqrt{10{,}100}$.

So $\dfrac{dz}{dt} = \dfrac{1}{z}\left[(x - 150)\dfrac{dx}{dt} + y\dfrac{dy}{dt}\right] = \dfrac{-10(35) + 100(25)}{\sqrt{10{,}100}} = \dfrac{215}{\sqrt{101}} \approx 21.4$ km/h.

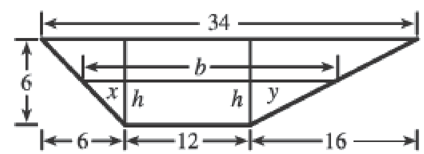
16. 

We are given that $\dfrac{dx}{dt} = 1.6$ m/s. By similar triangles, $\dfrac{y}{12} = \dfrac{2}{x} \Rightarrow y = \dfrac{24}{x} \Rightarrow$ $\dfrac{dy}{dt} = -\dfrac{24}{x^2}\dfrac{dx}{dt} = -\dfrac{24}{x^2}(1.6)$. When $x = 8$, $\dfrac{dy}{dt} = -\dfrac{24(1.6)}{64} = -0.6$ m/s, so the shadow is decreasing at a rate of 0.6 m/s.

20. 

Given $\dfrac{dy}{dt} = -1$ m/s, find $\dfrac{dx}{dt}$ when $x = 8$ m. $y^2 = x^2 + 1 \Rightarrow 2y\dfrac{dy}{dt} = 2x\dfrac{dx}{dt} \Rightarrow$ $\dfrac{dx}{dt} = \dfrac{y}{x}\dfrac{dy}{dt} = -\dfrac{y}{x}$. When $x = 8$, $y = \sqrt{65}$, so $\dfrac{dx}{dt} = -\dfrac{\sqrt{65}}{8}$. Thus, the boat approaches the dock at $\dfrac{\sqrt{65}}{8} \approx 1.01$ m/s.

26. The figure is drawn without the top 3 feet.



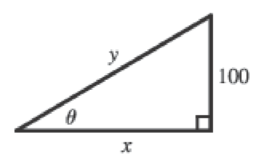
$V = \frac{1}{2}(b + 12)h(20) = 10(b + 12)h$ and, from similar triangles,

$\dfrac{x}{h} = \dfrac{6}{6}$ and $\dfrac{y}{h} = \dfrac{16}{6} = \dfrac{8}{3}$, so $b = x + 12 + y = h + 12 + \dfrac{8h}{3} = 12 + \dfrac{11h}{3}$.

Thus, $V = 10\left(24 + \dfrac{11h}{3}\right)h = 240h + \dfrac{110h^2}{3}$ and so $0.8 = \dfrac{dV}{dt} = \left(240 + \dfrac{220}{3}h\right)\dfrac{dh}{dt}$.

When $h = 5$, $\dfrac{dh}{dt} = \dfrac{0.8}{240 + 5(220/3)} = \dfrac{3}{2275} \approx 0.00132$ ft/min.

28. We are given $dx/dt = 8$ ft/s. $\cot\theta = \dfrac{x}{100} \Rightarrow x = 100\cot\theta \Rightarrow$



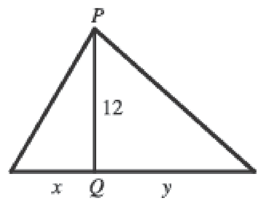
$\dfrac{dx}{dt} = -100\csc^2\theta\,\dfrac{d\theta}{dt} \Rightarrow \dfrac{d\theta}{dt} = -\dfrac{\sin^2\theta}{100}\cdot 8$. When $y = 200$, $\sin\theta = \dfrac{100}{200} = \dfrac{1}{2} \Rightarrow$

$\dfrac{d\theta}{dt} = -\dfrac{(1/2)^2}{100}\cdot 8 = -\dfrac{1}{50}$ rad/s. The angle is decreasing at a rate of $\frac{1}{50}$ rad/s.

34. $PV^{1.4} = C \Rightarrow P\cdot 1.4V^{0.4}\dfrac{dV}{dt} + V^{1.4}\dfrac{dP}{dt} = 0 \Rightarrow \dfrac{dV}{dt} = -\dfrac{V^{1.4}}{P\cdot 1.4V^{0.4}}\dfrac{dP}{dt} = -\dfrac{V}{1.4P}\dfrac{dP}{dt}$.

When $V = 400$, $P = 80$ and $\dfrac{dP}{dt} = -10$, so we have $\dfrac{dV}{dt} = -\dfrac{400}{1.4(80)}(-10) = \dfrac{250}{7}$. Thus, the volume is increasing at a rate of $\frac{250}{7} \approx 36$ cm^3/min.

38. Using Q for the origin, we are given $\frac{dx}{dt} = -2$ ft/s and need to find $\frac{dy}{dt}$ when $x = -5$.



Using the Pythagorean Theorem twice, we have $\sqrt{x^2+12^2}+\sqrt{y^2+12^2}=39$, the total length of the rope. Differentiating with respect to t, we get

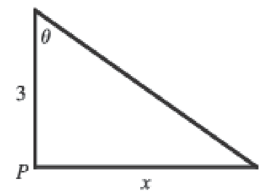
$$\frac{x}{\sqrt{x^2+12^2}}\frac{dx}{dt}+\frac{y}{\sqrt{y^2+12^2}}\frac{dy}{dt}=0, \text{ so } \frac{dy}{dt}=-\frac{x\sqrt{y^2+12^2}}{y\sqrt{x^2+12^2}}\frac{dx}{dt}.$$

Now when $x=-5$, $39=\sqrt{(-5)^2+12^2}+\sqrt{y^2+12^2}=13+\sqrt{y^2+12^2} \iff \sqrt{y^2+12^2}=26$, and

$y=\sqrt{26^2-12^2}=\sqrt{532}$. So when $x=-5$, $\frac{dy}{dt}=-\frac{(-5)(26)}{\sqrt{532}\,(13)}(-2)=-\frac{10}{\sqrt{133}}\approx -0.87$ ft/s.

So cart B is moving towards Q at about 0.87 ft/s.

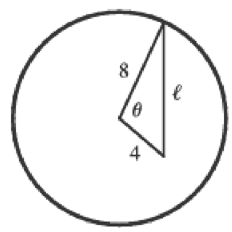
40. We are given that $\frac{d\theta}{dt}=4(2\pi)=8\pi$ rad/min. $x=3\tan\theta \Rightarrow$



$\frac{dx}{dt}=3\sec^2\theta\,\frac{d\theta}{dt}$. When $x=1$, $\tan\theta=\frac{1}{3}$, so $\sec^2\theta=1+\left(\frac{1}{3}\right)^2=\frac{10}{9}$

and $\frac{dx}{dt}=3\left(\frac{10}{9}\right)(8\pi)=\frac{80}{3}\pi\approx 83.8$ km/min.

46. The hour hand of a clock goes around once every 12 hours or, in radians per hour, $\frac{2\pi}{12}=\frac{\pi}{6}$ rad/h. The minute hand goes around once an hour, or at the rate of 2π rad/h. So the angle θ between them (measuring clockwise from the minute hand to the hour hand) is changing at the rate of $d\theta/dt=\frac{\pi}{6}-2\pi=-\frac{11\pi}{6}$ rad/h. Now, to relate θ to ℓ, we use the Law of Cosines: $\ell^2=4^2+8^2-2\cdot4\cdot8\cdot\cos\theta=80-64\cos\theta$ $(\star)$.

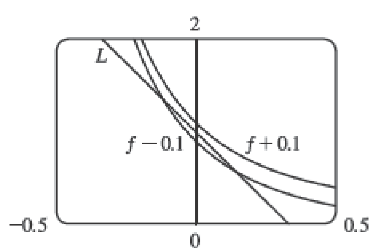


Differentiating implicitly with respect to t, we get $2\ell\frac{d\ell}{dt}=-64(-\sin\theta)\frac{d\theta}{dt}$. At 1:00, the angle between the two hands is one-twelfth of the circle, that is, $\frac{2\pi}{12}=\frac{\pi}{6}$ radians. We use $(\star)$ to find ℓ at 1:00: $\ell=\sqrt{80-64\cos\frac{\pi}{6}}=\sqrt{80-32\sqrt{3}}$.

Substituting, we get $2\ell\frac{d\ell}{dt}=64\sin\frac{\pi}{6}\left(-\frac{11\pi}{6}\right) \Rightarrow \frac{d\ell}{dt}=\frac{64\left(\frac{1}{2}\right)\left(-\frac{11\pi}{6}\right)}{2\sqrt{80-32\sqrt{3}}}=-\frac{88\pi}{3\sqrt{80-32\sqrt{3}}}\approx -18.6.$

So at 1:00, the distance between the tips of the hands is decreasing at a rate of 18.6 mm/h $\approx$ **0.005 mm/s**.

8. $f(x)=(1+x)^{-3} \Rightarrow f'(x)=-3(1+x)^{-4}$, so $f(0)=1$ and $f'(0)=-3$. Thus, $f(x)\approx f(0)+f'(0)(x-0)=1-3x$. We need $(1+x)^{-3}-0.1<1-3x<(1+x)^{-3}+0.1$, which is true when $-0.116<x<0.144$.



12. (a) For $y = f(s) = \dfrac{s}{1+2s}$, $f'(s) = \dfrac{(1+2s)(1) - s(2)}{(1+2s)^2} = \dfrac{1}{(1+2s)^2}$, so $dy = \dfrac{1}{(1+2s)^2}\,ds$.

(b) For $y = f(u) = e^{-u}\cos u$, $f'(u) = e^{-u}(-\sin u) + \cos u(-e^{-u}) = -e^{-u}(\sin u + \cos u)$, so

$dy = -e^{-u}(\sin u + \cos u)\,du$.

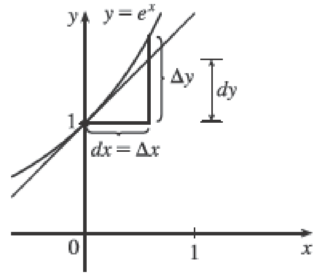
16. (a) $y = \cos\,\pi x \quad\Rightarrow\quad dy = -\sin\,\pi x \cdot \pi\,dx = -\pi\,\sin \pi x\,dx$

(b) $x = \frac{1}{3}$ and $dx = -0.02 \quad\Rightarrow\quad dy = -\pi\,\sin\frac{\pi}{3}(-0.02) = \pi\left(\sqrt{3}/2\right)(0.02) = 0.01\pi\,\sqrt{3} \approx 0.054$.

22. $y = f(x) = e^x$, $x = 0$, $\Delta x = 0.5 \quad\Rightarrow$

$\Delta y = f(0.5) - f(0) = \sqrt{e} - 1\;\;[\approx 0.65]$

$dy = e^x\,dx = e^0(0.5) = 0.5$



26. $y = f(x) = 1/x \quad\Rightarrow\quad dy = -1/x^2\,dx$. When $x = 4$ and $dx = 0.002$, $dy = -\frac{1}{16}(0.002) = -\frac{1}{8000}$, so

$\frac{1}{4.002} \approx f(4) + dy = \frac{1}{4} - \frac{1}{8000} = \frac{1999}{8000} = 0.249875$.

28. $y = f(x) = \sqrt{x} \quad\Rightarrow\quad dy = \dfrac{1}{2\sqrt{x}}\,dx$. When $x = 100$ and $dx = -0.2$, $dy = \dfrac{1}{2\sqrt{100}}(-0.2) = -0.01$, so

$\sqrt{99.8} = f(99.8) \approx f(100) + dy = 10 - 0.01 = 9.99$.

34. (a) $A = \pi r^2 \quad\Rightarrow\quad dA = 2\pi r\,dr$. When $r = 24$ and $dr = 0.2$, $dA = 2\pi(24)(0.2) = 9.6\pi$, so the maximum possible error in the calculated area of the disk is about $9.6\pi \approx 30\text{ cm}^2$.

(b) Relative error $= \dfrac{\Delta A}{A} \approx \dfrac{dA}{A} = \dfrac{2\pi r\,dr}{\pi r^2} = \dfrac{2\,dr}{r} = \dfrac{2(0.2)}{24} = \dfrac{0.2}{12} = \dfrac{1}{60} = 0.01\overline{6}$.

Percentage error $=$ relative error $\times 100\% = 0.01\overline{6} \times 100\% = 1.\overline{6}\%$.

36. For a hemispherical dome, $V = \frac{2}{3}\pi r^3 \quad\Rightarrow\quad dV = 2\pi r^2\,dr$. When $r = \frac{1}{2}(50) = 25$ m and $dr = 0.05$ cm $= 0.0005$ m,

$dV = 2\pi(25)^2(0.0005) = \frac{5\pi}{8}$, so the amount of paint needed is about $\frac{5\pi}{8} \approx 2\text{ m}^3$.