Periodica Mathematica Hungarica Vol. 17 (4), (1986), pp. 265—277

A REDUCTION OF KELLER'S CONJECTURE

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Abstract

A family of translates of a closed n-dimensional cube is called a cube tiling if the union of the cubes is the whole n-space and their interiors are disjoint. According to a famous unsolved conjecture of O. H. Keller, two of the cubes in an n-dimensional cube tiling must share a complete $(n - 1)$-dimensional face. In this paper we shall prove that to solve Keller's conjecture it is sufficient to examine certain factorizations of direct sum of finitely many cyclic group of order four.

I.

About fifty years ago O. H. Keller conjectured that in a family of the translates of an n-dimensional closed cube whose union is the whole n-dimensional space and whose interiors are disjoint some two cubes must have a common $(n - 1)$-dimensional face.

In [2] G. Hajós formulated this conjecture by means of factorizations of finite abelian groups. If G is an abelian group written additively and $H, A_1, \ldots, A_n$ are its subsets and each h in H is uniquely expressible in the form

$$h = a_1 + \ldots + a_n, \ a_1 \in A_1, \ldots, \ a_n \in A_n$$

and each sum $a_1 + \ldots + a_n$ is in H, then we write $H = A_1 + \ldots + A_n$ and call this equation a factorization of H by the subsets $A_1, \ldots, A_n$. If a subset of an abelian group has the form $\{0, g, 2g, \ldots, (q - 1)g\}$ and it differs from $\{0\}$, then we shall denote it by $[g]_q$. Throughout this paper we shall assume that q is not greater than the order of g. G. Hajós reduced Keller's conjecture to the following statement.

If G is a finite abelian group and

$$G = K + [g_1]_{q_1} + \ldots + [g_n]_{q_n}$$

AMS (MOS) subject classifications (1980). Primary 10E30; Secondary 20K01, 52A45.

Key words and phrases. Cube tiling, factorization of abelian groups.

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is one of its factorizations, then there exists i such that $1 \leq i \leq n$ and $q_i g_i \in K - K$, where $K - K = \{k - k' \colon k,\ k' \in K\}$.

But this problem has been remaining unsolved in general case. In [5] K. Seitz has obtained a solution for case when G is a cyclic group of a prime power order and in [4] A. D. Sands proved the above statement for case when G is a cyclic group whose order is a product of two prime powers.

In this paper we give a new formulation of Keller's conjecture, which allows us to verify the criterion of Hajós' type for more special groups:

THEOREM. *Keller's conjecture is equivalent to the following statement:*

If G is an internal direct sum of cyclic groups generated by the elements $g_1, g_2, \dots, g_m$, respectively, each of them is of order 4, *and if*

$$G = K \dot{+} [g_1]_2 + \dots + [g_m]_2$$

is a factorization, then always there exists a generator g_i $(1 \leq i \leq m)$ such that

$$2g_i \in K - K = \{k - k' \colon k, k' \in K\}.$$

The proof will be an improvement of the way of Hajós. It is based on Propositions 1, 2 which unable us to transform a cube tiling, giving counter-example for Keller's conjecture, onto such another cube tiling with periodic structure. Later on we specialize this periodicity to get groups mentioned in the theorem.

II.

Let $\mathscr{E}^n$, R and Z denote the n-dimensional euclidean space, the real number field and the integer number ring, respectively. Translations of $\mathscr{E}^n$ belong to the n-dimensional vector space $\mathbf{E}^n$ over the real number field R. Elements of $\mathbf{E}^n$ form an abelian group with respect to the addition of vectors so we can speak about factorizations of a subset of $\mathbf{E}^n$.

Let H be a subset of an abelian group and let g be a nonzero element of this abelian group. If $H = H + g$, then H is said to be a periodic set with period g. Let $\mathbf{e}_1, \dots, \mathbf{e}_n$ be a fixed orthonormed basis in $\mathbf{E}^n$ and denote by $\mathcal{C}_P$ that n-dimensional closed unit cube whose center is P and whose edges are parallel to the coordinate unit vectors $\mathbf{e}_1, \dots, \mathbf{e}_n$.

Obviously, cubes $\mathcal{C}_P$ and $\mathcal{C}_Q$ have a common $(n-1)$-dimensional face if and only if either $\overrightarrow{PQ} = \mathbf{e}_i$ or $\overrightarrow{PQ} = -\mathbf{e}_i$ for some i, $1 \leq i \leq n$. Since notations P and Q can be interchanged we may always restrict our investigations to case $\overrightarrow{PQ} = \mathbf{e}_i$.

The section of interiors of cubes $\mathcal{C}_P$ and $\mathcal{C}_Q$ is nonempty if and only if $|a_1| < 1, \dots, |a_n| < 1$ hold, where $\overrightarrow{PQ} = a_1\mathbf{e}_1 + \dots + a_n\mathbf{e}_n$. The interior of a set $\mathcal{A} \subseteq \mathcal{E}^n$ is denoted by $\operatorname{int} \mathcal{A}$.

Let O be a fixed point of $\mathcal{E}^n$. If $\mathbf{L}$ is a subset of $\mathbf{E}^n$, then the set of cubes $\{\mathcal{C}_P\colon \overrightarrow{OP} \in \mathbf{L}\}$ we shall shortly denote by $(\mathcal{C}_O, \mathbf{L})$. The cube system $(\mathcal{C}_O, \mathbf{L})$ is called a tiling if the union of its point sets, denoted by $\bigcup (\mathcal{C}_O, \mathbf{L})$, is $\mathcal{E}^n$ and the interiors of the cubes are disjoint. Obviously the set $\{P\colon \overrightarrow{OP} \in \mathbf{L}\}$ has not any accumulation points, if $(\mathcal{C}_0, \mathbf{L})$ forms a tiling.

The subset $\mathbf{L}$ of $\mathbf{E}^n$ is called a lattice if $\mathbf{L}$ is a nonzero subgroup of $\mathbf{E}^n$ and the point set $\{P\colon \overrightarrow{OP} \in \mathbf{L}\}$ has not any accumulation points. Clearly, if $\mathbf{l}_1, \dots, \mathbf{l}_s$ are linearly independent vectors in $\mathbf{E}^n$, then the vector set

$$\{z_1\mathbf{l}_1 + \dots + z_s\mathbf{l}_s\colon z_1, \dots, z_s \in Z\}$$

is a lattice. In this case we shall say that this lattice is spanned by the vectors $\mathbf{l}_1, \dots, \mathbf{l}_s$.

III.

If $\mathbf{L}$ is a subset of $\mathbf{E}^n$ and i is a given number $1 \le i \le n$, then the elements of $\mathbf{L}$ can be divided into disjoint subsets $\mathbf{X}_{u,v}^{(i)}$ in the following way: if the integer part and the fractional part of the i-th coordinate of $\mathbf{l} \in \mathbf{L}$ are u and v, respectively, then let $\mathbf{l} \in \mathbf{X}_{u,v}^{(i)}$. (For example, the integer parts and the fractional parts of 3,2 and $-5{,}1$ are 3, 0,2 and -6, 0,9, respectively.)

PROPOSITION 1. *If $(\mathcal{C}_O, \mathbf{L})$ is tiling, then, for any fixed v $(0 \le v < 1)$,*

$$\bigcup (\mathcal{C}_O, \mathbf{X}_{u,v}^{(i)}) = \bigcup (\mathcal{C}_O, \mathbf{X}_{r,v}^{(i)} + (u - r)\mathbf{e}_i)$$

holds for each $u, r \in Z$ and i, $1 \le i \le n$. Saying in words, the point set, constituting the cubes in $(\mathcal{C}_O, \mathbf{L})$ whose i-th coordinate has a fixed fractional part v, is mapped onto itself by arbitrary integer translation in direction $\mathbf{e}_i$.

PROOF. For simplicity we assume that $i = 1$. We shall prove induction by n. Since the section of an n-dimensional cube and an $(n - 1)$-dimensional plane, which is parallel to one of the $(n - 1)$-dimensional faces of the cube, is an $(n - 1)$-dimensional cube, it seems to be a good idea to consider the intersection of $(\mathcal{C}_O, \mathbf{L})$ and an $(n - 1)$-dimensional plane. Obviously, the union of $(n - 1)$-dimensional cubes obtained in this intersection will be the whole $(n - 1)$-dimensional plane. But these $(n - 1)$-dimensional cubes could have nonempty common interiors. Thus we have to use a finer method than this.

Let $\mathcal{S}$ be any $(n - 1)$-dimensional planes which are perpendicular to $\mathbf{e}_2$. The hyperplane $\mathcal{S}$ divides $\mathcal{E}^n$ into two half spaces. One of them contains the ending point of $\mathbf{e}_2$ when its starting point is in $\mathcal{S}$. Denote by $\mathcal{F}$ this half space. We state

$$(\overline{\mathcal{C}}_{\bar{O}}, \mathbf{L}) = \{\overline{\mathcal{C}}_{\bar{P}}\colon \overline{\mathcal{C}}_{\bar{P}} = \mathcal{C}_P \cap \mathcal{S},\ \mathcal{C}_P \cap \operatorname{int} \mathcal{F} \neq \emptyset,\ \overrightarrow{OP} \in \mathbf{L}\}$$

is a tiling in $\mathcal{S}$.

Indeed, assume the contrary, i.e. that $\operatorname{int} \overline{\mathcal{C}}_{\bar{P}} \cap \operatorname{int} \overline{\mathcal{C}}_{\bar{Q}} \neq \emptyset$. If

$$\overrightarrow{\bar{P}\bar{Q}} = \sum_{\substack{s=1 \\ s\neq 2}}^{n} a_s \mathbf{e}_s,$$

then here each $|a_s| < 1$. Since $\overrightarrow{PQ} = \overrightarrow{P\bar{P}} + \overrightarrow{\bar{P}\bar{Q}} + \overrightarrow{\bar{Q}Q}$ and $\overrightarrow{P\bar{P}} = \mu \mathbf{e}_2$, $\overrightarrow{Q\bar{Q}} = \nu \mathbf{e}_2$, where from $\mathcal{C}_P \cap \operatorname{int} \mathcal{F} \neq \emptyset$ and $\mathcal{C}_Q \cap \operatorname{int} \mathcal{F} \neq \emptyset$ we have $-1/2 \leq \mu, \nu < 1/2$,

$$\overrightarrow{PQ} = (\mu - \nu)\mathbf{e}_2 + \sum_{\substack{s=1 \\ s\neq 2}}^{n} a_s \mathbf{e}_s.$$

Now $|\mu - \nu| < 1$ and $|a_s| < 1$, so $\operatorname{int} \mathcal{C}_P \cap \operatorname{int} \mathcal{C}_Q \neq \emptyset$ which is a contradiction.

It has remained only to prove that if $S \in \mathcal{S}$, then S is a point of some $\mathcal{C}_P$. Let $\mathcal{G}$ be a bounded point set in $\operatorname{int} \mathcal{F}$ which has the only accumulation point S. Since $(\mathcal{C}_O, \mathbf{L})$ is a tiling, so the set $\mathcal{G}$ is contained in finitely many cubes of system $(\mathcal{C}_O, \mathbf{L})$, hence there exists a $\mathcal{C}_P \in (\mathcal{C}_O, \mathbf{L})$ such that $\mathcal{C}_P$ has infinitely many elements of the set $\mathcal{G}$. Obviously, $\mathcal{C}_P \cap \operatorname{int} \mathcal{F} \neq \emptyset$. Since the point S is the accumulation point of each infinite subset of $\mathcal{G}$ and since $\mathcal{C}_P$ is a compact set, hence $S \in \mathcal{C}_P$.

Now the induction follows. The statement of Proposition 1 is trivial in case $n = 1$. Assume that $n > 1$ and the statement holds in $n - 1$ dimensions. Consider the tilings $(\mathcal{C}_O, \mathbf{L})$ and $(\overline{\mathcal{C}}_{\bar{O}}, \overline{\mathbf{L}})$ as above for any hyperplane $\mathcal{S}$ perpendicular to $\mathbf{e}_2$. Divide the elements of $\overline{\mathbf{L}}$ into disjoint subsets $\overline{\mathbf{X}}^{(1)}_{u,v}$ in the following way; if the integer part and the fractional part of the first coordinate of $\bar{\mathbf{l}} \in \overline{\mathbf{L}}$ are u and v, respectively, then let $\bar{\mathbf{l}} \in \overline{\mathbf{X}}^{(1)}_{u,v}$. By the inductive assumption for any fixed v $(0 \leq v < 1)$

$$\bigcup (\overline{\mathcal{C}}_{\bar{O}}, \overline{\mathbf{X}}^{(1)}_{u,v}) = \bigcup (\overline{\mathcal{C}}_{\bar{O}}, \overline{\mathbf{X}}^{(1)}_{r,v} + (u - r)\mathbf{e}_1)$$

holds $(u, r \in Z)$ and note that

$$(\overline{\mathcal{C}}_{\bar{O}}, \overline{\mathbf{X}}^{(1)}_{u,v}) = \{\overline{\mathcal{C}}_{\bar{P}}\colon \overline{\mathcal{C}}_{\bar{P}} = \mathcal{C}_P \cap \mathcal{S},\ \mathcal{C}_P \cap \operatorname{int} \mathcal{F} \neq \emptyset,\ \overrightarrow{OP} \in \mathbf{X}^{(1)}_{u,v}\}.$$

So finally we conclude that

$$\bigcup (\mathcal{C}_O, \mathbf{X}^{(1)}_{u,v}) = \bigcup (\bigcup_{\mathcal{S}} (\overline{\mathcal{C}}_{\bar{O}}, \mathbf{X}^{(1)}_{u,v})) = \bigcup (\bigcup_{\mathcal{S}} (\overline{\mathcal{C}}_{\bar{O}}, \overline{\mathbf{X}}^{(1)}_{r,v} + (r - u)\mathbf{e}_1)) =$$
$$= \bigcup (\mathcal{C}_O, \mathbf{X}^{(1)}_{r,v} + (r - u)\mathbf{e}_1)$$

where $\bigcup_{\mathcal{S}}$ denotes the union of the point sets in $\mathcal{S}$ considered, if $\mathcal{S}$ runs over the set of hyperplanes perpendicular to $\mathbf{e}_2$. This completes the proof.

PROPOSITION 2. *If* $(\mathcal{C}_O, \mathbf{L})$ *is a tiling, then*

$$\bigcup (\mathcal{C}_O, \bigcup_u \mathbf{X}^{(i)}_{u,v}) = \bigcup (\mathcal{C}_O, \bigcup \mathbf{X}^{(i)}_{u,v} + \alpha \mathbf{e}_i)$$

for each $1 \leq i \leq n$ *and* $\alpha \in R$.

The PROOF is carried out analogously to the proof of the preceding proposition. We say: the point set constituting all cubes in $(\mathcal{C}_O, \mathbf{L})$ with any fixed i-th fractional part v $(0 \leq v < 1)$ is mapped onto itself by any translation of direction $\mathbf{e}_i$. This assertion is due to Hajós [1].

PROPOSITION 3. *If there is a counter-example* $(\mathcal{C}_O, \mathbf{L})$ *for Keller's conjecture, then there is a counter-example* $(\mathcal{C}_O, \mathbf{L}')$ *such that* $\mathbf{L}'$ *is a periodic set with periods* $2\mathbf{e}_1, \ldots, 2\mathbf{e}_n$.

PROOF. Divide the elements of $\mathbf{L}$ into disjoint subsets $\mathbf{X}^{(1)}_{u,v}$ in the previous way and consider the set $\mathbf{L}_1$ given by

$$\mathbf{L}_1 = \bigcup_{u \in Z} (\bigcup (\mathbf{X}^{(1)}_{0,v} \cup \mathbf{X}^{(1)}_{1,v} + 2u\mathbf{e}_1).$$

We shall prove that the system $(\mathcal{C}_O, \mathbf{L}_1)$ is a tiling and in it no two cubes have a common $(n-1)$-dimensional face. Using the fact

$$\bigcup (\mathcal{C}_O, \mathbf{X}^{(1)}_{u,v}) = \bigcup (\mathcal{C}_O, \mathbf{X}^{(1)}_{0,v} + u\mathbf{e}_1)$$

by Proposition 1, we have

$$\begin{aligned}
\mathcal{E}^n = \bigcup (\mathcal{C}_O, \mathbf{L}) = \bigcup (\mathcal{C}_O, \bigcup_u \bigcup_v \mathbf{X}^{(1)}_{u,v}) = \bigcup_u \bigcup_v \bigcup (\mathcal{C}_O, \mathbf{X}^{(1)}_{u,v}) = \\
= [\bigcup_{2|u} \bigcup_v \bigcup (\mathcal{C}_O, \mathbf{X}^{(1)}_{u,v})] \cup [\bigcup_{2\nmid u} \bigcup_v \bigcup (\mathcal{C}_O, \mathbf{X}^{(1)}_{u,v})] = \\
= [\bigcup_{2|u} \bigcup_v \bigcup (\mathcal{C}_O, \mathbf{X}^{(1)}_{0,v} + u\mathbf{e}_1)] \cup [\bigcup_{2|u} \bigcup_v \bigcup (\mathcal{C}_O, \mathbf{X}^{(1)}_{1,v} + u\mathbf{e}_1)] = \\
= \bigcup_u \bigcup_v \bigcup (\mathcal{C}_O, (\mathbf{X}^{(1)}_{0,v} \cup \mathbf{X}^{(1)}_{1,v}) + 2u\mathbf{e}_1) = \\
= \bigcup (\mathcal{C}_O, \bigcup_u \bigcup_v (\mathbf{X}^{(1)}_{0,v} \cup \mathbf{X}^{(1)}_{1,v}) + 2u\mathbf{e}_1) = \bigcup (\mathcal{C}_O, \mathbf{X}_1).
\end{aligned}$$

Assume the contrary, i.e. that $\operatorname{int} \mathcal{C}_P \cap \operatorname{int} \mathcal{C}_Q \neq \emptyset$ for $\mathcal{C}_P, C_Q \in (\mathcal{C}_O, \mathbf{L}_1)$. In other words assume that $\overrightarrow{OP}, \overrightarrow{OQ} \in \mathbf{L}_1$ and $|a_1| < 1, \ldots, |a_n| < 1$, where $\overrightarrow{PQ} = a_1\mathbf{e}_1 + \ldots a_n\mathbf{e}_n$. Let $\overrightarrow{OP} \in \mathbf{X}^{(1)}_{i,v} + 2r\mathbf{e}_1$ and $\overrightarrow{OQ} \in \mathbf{X}^{(1)}_{j,w} + 2s\mathbf{e}_1$. Hence

$$\overrightarrow{PQ} \in (\mathbf{X}^{(1)}_{i,w} - \mathbf{X}^{(1)}_{i,v}) + 2(s-r)\mathbf{e}_1,$$

where $i, j \in \{0, 1\}$. Since

$$\bigcup (\mathcal{C}_O, \mathbf{X}_{i,v}^{(1)} + 2r\mathbf{e}_1) = \bigcup (\mathcal{C}_O, \mathbf{X}_{2r+i,v}^{(1)}), \ \bigcup (\mathcal{C}_O, \mathbf{X}_{j,w}^{(1)} + 2s\mathbf{e}_1) = \bigcup (\mathcal{C}_O, \mathbf{X}_{2s+j,w}^{(1)})$$

and

$$[\operatorname{int} \bigcup (\mathcal{C}_O, \mathbf{X}_{2r+i,v}^{(1)}) \cap \operatorname{int} \bigcup (\mathcal{C}_O, \mathbf{X}_{2s+j,w}^{(1)})] = \emptyset$$

if $v \neq w$, moreover, $\mathcal{C}_P \in (\mathcal{C}_O, \mathbf{X}_{i,v}^{(1)} + 2r\mathbf{e}_1)$, $\mathcal{C}_Q \in (\mathcal{C}_O, \mathbf{X}_{j,w}^{(1)} + 2s\mathbf{e}_1)$, from the assumption it follows $v = w$. Then from $a_1 = (j - i) + (w - v) + 2(s - r)$ and $i, j \in \{0, 1\}$, $r, s \in Z$, $|a_1| < 1$ we conclude that $s = r$ and therefore

$$\overrightarrow{PQ} \in \mathbf{X}_{j,w}^{(1)} - \mathbf{X}_{j,v}^{(1)} \subset \mathbf{L} - \mathbf{L}$$

which violates the fact $(\mathcal{C}_O, \mathbf{L})$ was a tiling. Thus $(\mathcal{C}_O, \mathbf{L}_1)$ is a tiling.

Now assume the contrary, i.e., that $(\mathcal{C}_O, \mathbf{L}_1)$ has two cubes $\mathcal{C}_P, \mathcal{C}_Q$ sharing a complete $(n-1)$-dimensional face. Let $\overrightarrow{PQ} = \mathbf{e}_t$ and $\overrightarrow{OP} \in \mathbf{X}_{i,v}^{(1)} + 2r\mathbf{e}_1$, $\overrightarrow{OQ} \in \mathbf{X}_{j,w}^{(1)} + 2s\mathbf{e}_1$, where $i, j \in \{0, 1\}$ so

$$\overrightarrow{PQ} = \mathbf{e}_t \in \mathbf{X}_{j,w}^{(1)} - (\mathbf{X}_{i,v}^{(1)} + 2(r - s)\mathbf{e}_1).$$

In case $t \neq 1$, from $0 = (j - i) + (w - v) + 2(s - r)$ and $i, j \in \{0, 1\}$, $r, s \in Z$, $|w - v| < 1$ we have $v = w$ and $r = s$, that is

$$\overrightarrow{PQ} = \mathbf{e}_t \in \mathbf{X}_{j,w}^{(1)} - \mathbf{X}_{i,v}^{(1)} \subset \mathbf{L} - \mathbf{L}$$

which is a contradiction since $(\mathcal{C}_O, \mathbf{L})$ was a counter-example for Keller's conjecture. In case $t = 1$, from $1 = (j - i) + (w - v) + 2(s - r)$ and $i, j \in \{0, 1\}$ $r, s \in Z$, $|w - v| < 1$ we have $v = w$ and either $r = s$ or $s = r + 1$. Therefore, either

$$\overrightarrow{PQ} = \mathbf{e}_1 \in (\mathbf{X}_{j,v}^{(1)} - \mathbf{X}_{i,v}^{(1)}) \subset \mathbf{L} - \mathbf{L}$$

or

$$\overrightarrow{PQ} = \mathbf{e}_1 \in (\mathbf{X}_{j,v}^{(1)} - (\mathbf{X}_{i,v}^{(1)} + 2\mathbf{e}_1),$$

that is

$$- \mathbf{e}_1 \in (\mathbf{X}_{j,v}^{(1)} - \mathbf{X}_{i,v}^{(1)}) \subset \mathbf{L} - \mathbf{L}$$

are contradictions, respectively.

The set $\mathbf{L}_1$ is periodic with period $2\mathbf{e}_1$ because

$$\mathbf{L}_1 = \bigcup_u (\bigcup_v (\mathbf{X}_{0,v}^{(1)} \cup \mathbf{X}_{1,v}^{(1)}) + 2u\mathbf{e}_1) = \bigcup_u (\bigcup_v (\mathbf{X}_{0,v}^{(1)} \cup \mathbf{X}_{1,v}^{(1)}) + 2(u+1)\mathbf{e}_1) =$$

$$= [\bigcup_u (\bigcup_v (\mathbf{X}_{0,v}^{(1)} \cup \mathbf{X}_{1,v}^{(1)}) + 2u\mathbf{e}_1)] + 2\mathbf{e}_1 = \mathbf{L}_1 + 2\mathbf{e}_1.$$

Divide elements of $\mathbf{L}_1$ into disjoint subsets $\mathbf{X}_{u,v}^{(2)}$ in the previous wa and consider the set $\mathbf{L}_2$ defined by

$$\mathbf{L}_2 = \bigcup_u (\bigcup_v (\mathbf{X}_{0,v}^{(2)} \cup \mathbf{X}_{1,v}^{(2)}) + 2u\mathbf{e}_2).$$

We can check analogously that the system $(\mathcal{C}_O, \mathbf{L}_2)$ is a tiling in which no two cubes have a common $(n-1)$-dimensional face and $\mathbf{L}_2$ is a periodic set with period $2\mathbf{e}_2$. Now we shall prove that $\mathbf{L}_2$ is a periodic set with $2\mathbf{e}_1$ as well. From

$$\bigcup_u \bigcup_v \mathbf{X}_{u,v}^{(2)} = \mathbf{L}_1 = \mathbf{L}_1 + 2\mathbf{e}_1 = \bigcup_u \bigcup_v \mathbf{X}_{u,v}^{(2)} + 2\mathbf{e}_1$$

and from that the second coordinates of vectors of $\mathbf{X}_{u,v}^{(2)}$ and of $\mathbf{X}_{u,v}^{(2)} + 2\mathbf{e}_1$ are the same, respectively, we conclude that $\mathbf{X}_{u,v}^{(2)}$ is a periodic set with period $2\mathbf{e}_1$ that is $\mathbf{X}_{u,v}^{(2)} = \mathbf{X}_{u,v}^{(2)} + 2\mathbf{e}_1$. Using this fact we have

$$\mathbf{L}_2 = \bigcup_u [\bigcup_v (\mathbf{X}_{0,v}^{(2)} \cup \mathbf{X}_{1,v}^{(2)}) + 2u\mathbf{e}_2] = \bigcup_u [\bigcup_v (\mathbf{X}_{0,v}^{(2)} + 2\mathbf{e}_1 \cup \mathbf{X}_{1,v}^{(2)} + 2\mathbf{e}_1) + 2u\mathbf{e}_2] =$$

$$= \bigcup_u [\bigcup_v (\mathbf{X}_{0,v}^{(2)} \cup \mathbf{X}_{1,v}^{(2)}) + 2u\mathbf{e}_2] + 2\mathbf{e}_1 = \mathbf{L}_2 + 2\mathbf{e}_1.$$

Divide the elements of $\mathbf{L}_2$ into disjoint subsets $\mathbf{X}_{u,v}^{(3)}$ as earlier and construct the set

$$\mathbf{L}_3 = \bigcup_u [\bigcup_v (\mathbf{X}_{0,v}^{(3)} \mathbf{X}_{1,v}^{(3)}) + 2u\mathbf{e}_3].$$

The system $(\mathcal{C}_O, \mathbf{L}_3)$ will be a tiling in which no two cubes having a common $(n-1)$-dimensional face. Step by step we can prove that $\mathbf{L}_3$ is a periodic set with periods $2\mathbf{e}_3$, $2\mathbf{e}_2$ and $2\mathbf{e}_1$. Continuing this process we finally have a system $(\mathcal{C}_O, \mathbf{L}_n) = (\mathcal{C}_O, \mathbf{L}')$ which is a counter-example for Keller's conjecture and $\mathbf{L}_n = \mathbf{L}'$ is a periodic set with periods $2\mathbf{e}_n, \ldots, 2\mathbf{e}_1$.

PROPOSITION 4. *If there is a counter-example for Keller's conjecture* $(\mathcal{C}_O, \mathbf{L})$, *then there is a counter-example for Keller's conjecture* $(\mathcal{C}_O, \mathbf{L}')$ *such that* $\mathbf{L}'$, *is a periodic set with periods* $2\mathbf{e}_1, \ldots, 2\mathbf{e}_n$ *and every vector of* $\mathbf{L}'$ *has rational coordinates such that each of their denominators is a power of two.*

PROOF. According to the preceding proposition we may assume $\mathbf{L}$ to be a periodic set with periods $2\mathbf{e}_1, \ldots, 2\mathbf{e}_n$. Let $\hat{\mathbf{L}}$ be the lattice spanned by the vectors $2\mathbf{e}_1, \ldots, 2\mathbf{e}_n$ and let us define the "cube"

$$\mathbf{C} = \{c_1\mathbf{e}_1 + \ldots + c_n\mathbf{e}_n\colon\ 0 \leq c_i < 2,\ 1 \leq i \leq n\}.$$

Since each $a_i \in R$ is uniquely expressible in the form $a_i = 2b_i + c_i$, where $b_i \in Z$ and $0 \leq c_i < 2$, hence each $\mathbf{l} = a_1\mathbf{e}_1 + \ldots + a_n\mathbf{e}_n \in \mathbf{L}$ is uniquely expressible in the form $\mathbf{l} = \mathbf{l}' + \hat{\mathbf{l}}$, where $\mathbf{l}' \in \mathbf{L} \cap \mathbf{C}$ and $\hat{\mathbf{l}} \in \hat{\mathbf{L}}$. Thus $\mathbf{L} = (\mathbf{L} \cap \mathbf{C}) + \hat{\mathbf{L}}$ is a factorization.

Divide the vectors of $\mathbf{L}$ into disjoint subsets $\mathbf{X}_{u,v}^{(1)}$ in the usual way. Now v runs over finitely many distinct values, since $\mathbf{L} \cap \mathbf{C}$ has finitely many elements, because $\mathbf{L}$ has not any accumulation point.

Consider now a new set $\mathbf{L}_1$ defined by

$$\mathbf{L}_1 = \bigcup_v \bigcup_u \mathbf{X}_{u,v}^{(1)} + \alpha_v^{(1)}\mathbf{e}_1,$$

where $(\alpha_v^{(1)} + v)$'s are distinct rational numbers such that $0 \leq \alpha_v^{(1)} + v < 1$ and their denominators are power of two.

We shall prove that the system $(\mathcal{C}_O, \mathbf{L}_1)$ is a counter example for Keller's conjecture. Using the fact that for any fixed v

$$\bigcup (\mathcal{C}_O, \bigcup_u \mathbf{X}_{u,v}^{(1)}) = \bigcup (\mathcal{C}_O, \bigcup \mathbf{X}_{u,v}^{(1)} + \alpha \mathbf{e}_1)$$

holds for each $\alpha \in R$ (see Proposition 2), we can easily prove, as earlier, that $(\mathcal{C}_O, \mathbf{L}_1)$ is a tiling

$$\mathcal{E}^n = \bigcup (\mathcal{C}_O, \mathbf{L}) = \bigcup (\mathcal{C}_O, \bigcup_u \bigcup_v \mathbf{X}_{u,v}^{(1)}) = \bigcup_v \bigcup (\mathcal{C}_O, \bigcup \mathbf{X}_{u,v}^{(1)}) =$$

$$= \bigcup_v \bigcup (\mathcal{C}_O, \bigcup_u \mathbf{X}_{u,v}^{(1)} + \alpha_v^{(1)} \mathbf{e}_1) = \bigcup (\mathcal{C}_O, \bigcup_v \bigcup_u \mathbf{X}_{u,v}^{(1)} + \alpha_v^{(1)} \mathbf{e}_1) = \bigcup (\mathcal{C}_O, \mathbf{L}_1).$$

Assume the contrary, i.e. that

$$\operatorname{int} \mathcal{C}_P \cap \operatorname{int} \mathcal{C}_Q \neq \emptyset \text{ for } \mathcal{C}_P, \mathcal{C}_Q \in (\mathcal{C}_O, \mathbf{L}_1).$$

This means that $\overrightarrow{OP}, \overrightarrow{OQ} \in \mathbf{L}_1$ and $|a_1| < 1, \ldots, |a_n| < 1$, where $\overrightarrow{PQ} = a_1\mathbf{e}_1 + \ldots + a_n\mathbf{e}_n$. Let

$$\overrightarrow{OP} \in \mathbf{X}_{u,v}^{(1)} + \alpha_v^{(1)}\mathbf{e}_1, \quad \overrightarrow{OQ} \in \mathbf{X}_{r,v}^{(1)} + \alpha_w^{(1)}\mathbf{e}_1.$$

Since

$$\bigcup (\mathcal{C}_O, \bigcup_u \mathbf{X}_{u,v}^{(1)}) = \bigcup (\mathcal{C}_O, \bigcup_u \mathbf{X}_{u,v}^{(1)} + \alpha_v^{(1)}\mathbf{e}_1)$$

$$\bigcup (\mathcal{C}_O, \bigcup_u \mathbf{X}_{u,w}^{(1)}) = \bigcup (\mathcal{C}_O, \bigcup_u \mathbf{X}_{u,w}^{(1)} + \alpha_w^{(1)}\mathbf{e}_1)$$

and

$$[\operatorname{int} \bigcup (\mathcal{C}_O, \bigcup_u X_{u,v}^{(1)})] \cap [\operatorname{int} \bigcup (\mathcal{C}_O, \bigcup_u X_{u,w}^{(1)})] = \emptyset$$

for $v \neq w$, moreover,

$$\mathcal{C}_P \in (\mathcal{C}_O, \bigcup_u \mathbf{X}_{u,v}^{(1)} + \alpha_v^{(1)}\mathbf{e}_1)$$

$$\mathcal{C}_Q \in (\mathcal{C}_O, \bigcup_u \mathbf{X}_{u,w}^{(1)} + \alpha_w^{(1)}\mathbf{e}_1),$$

from the assumption it follows that $v = w$. Then from

$$a_1 = (r - u) + (w + \alpha_w^{(1)}) - (v + \alpha_v^{(1)})$$

and from $|a_1| < 1$ we have $u = r$ hence $\overrightarrow{PQ} \in (\mathbf{X}_{u,v}^{(1)} - \mathbf{X}_{u,v}^{(1)}) \subset \mathbf{L} - \mathbf{L}$ which is a contradiction since $(\mathcal{C}_O, \mathbf{L})$ is a tiling. Consequently, $(\mathcal{C}_O, \mathbf{L}_1)$ is a tiling.

Now assume the contrary, i.e. $(\mathcal{C}_O, \mathbf{L}_1)$ has two cubes $\mathcal{C}_P$, $\mathcal{C}_Q$ having a common complete $(n-1)$-dimensional face. Let $\overrightarrow{PQ} = \mathbf{e}_t$ and

$$\overrightarrow{OP} \in \mathbf{X}_{u,v}^{(1)} + \alpha_v^{(1)}\mathbf{e}_1, \quad \overrightarrow{OQ} \in \mathbf{X}_{r,w}^{(1)} + \alpha_w^{(1)}\mathbf{e}_1.$$

From these we conclude that the first coordinate of $\overrightarrow{PQ}$, which is an integer number, is

$$(r-u)+(w+\alpha_w^{(1)})-(v+\alpha_v^{(1)})$$

and therefore we have $v=w$ and $\alpha_v^{(1)}=\alpha_w^{(1)}$. Thus

$$\overrightarrow{PQ}\in(\bigcup_u \mathbf{X}_{u,v}^{(1)})-(\bigcup_u \mathbf{X}_{u,v}^{(1)})\subset \mathbf{L}-\mathbf{L}$$

which violates the fact $(\mathcal{C}_O,\mathbf{L})$ was a counter-example for Keller's conjecture. Thus $(\mathcal{C}_O,\mathbf{L}_1)$ is a counter-example for Keller's conjecture and it is obvious that $\mathbf{L}_1$ is periodic with periods $2\mathbf{e}_1,\ldots,2\mathbf{e}_n$.

Divide the elements of $\mathbf{L}_1$ into disjoint subsets $\mathbf{X}_{u,v}^{(2)}$ in the usual way and consider the new set $\mathbf{L}_2$ given by

$$\mathbf{L}_2=\bigcup_v\,[\bigcup_u \mathbf{X}_{u,v}^{(2)}+\alpha_v^{(2)}\mathbf{e}_2].$$

where the $(\alpha_v^{(2)}+v)$'s are distinct rational numbers such that $0\le\alpha_v^{(2)}+v<1$ and their denominators are powers of two. The system $(\mathcal{C}_O,\mathbf{L}_2)$ will be a counter-example for Keller's conjecture. Continuing this process, we finally have the system $(\mathcal{C}_O,\mathbf{L}_n)=(\mathcal{C}_O,\mathbf{L}')$ which is a counter-example for Keller's conjecture. We can verify that each vector in $\mathbf{L}_n=\mathbf{L}'$ has rational coordinates whose denominators are powers of two and $\mathbf{L}_n=\mathbf{L}'$ is a periodic set with periods $2\mathbf{e}_1,\ldots,2\mathbf{e}_n$.

PROPOSITION 5. *If there is a counter-example for Keller's conjecture, then there exists a finite abelian group G and its factorization.*

$$G=K+[g_1']_2+\ldots+[g_m']_2$$

such that $2g_i'\notin K-K$ for each i, $1\le i\le m$.

PROOF. Let $(\mathcal{C}_O,\mathbf{L})$ be a counter-example for Keller's conjecture. We may assume that $\mathbf{L}$ is a periodic set with periods $2\mathbf{e}_1,\ldots,2\mathbf{e}_n$ and each vector in $\mathbf{L}$ has rational coordinates whose denominators are powers of two.

Denote $\acute{\mathbf{L}}$ the lattice spanned by $2\mathbf{e}_1,\ldots,2\mathbf{e}_n$ and introduce

$$\mathbf{C}=\{c_1\mathbf{e}_1+\ldots+c_n\mathbf{e}_n\colon\ 0\le c_i<2,\ 1\le i\le n\}$$

as previously. We have already known that the equation $\mathbf{L}=(\mathbf{L}\cap\mathbf{C})+\acute{\mathbf{L}}$ is a factorization. Since $\mathbf{L}\cap\mathbf{C}$ is a finite set, there exists a power of two, say q, which is the common multiple of denominators of coordinates of vectors of $\mathbf{L}\cap\mathbf{C}$. Denote $\tilde{\mathbf{L}}$ the lattice spanned by the vectors $(1/q)\mathbf{e}_1,\ldots,(1/q)\mathbf{e}_n$. Since $(\mathcal{C}_O,\mathbf{L})$ is a tiling and $\mathbf{L}\subset\tilde{\mathbf{L}}$

$$\tilde{\mathbf{L}}=[(1/q)\mathbf{e}_1]_q+\ldots+[(1/q)\mathbf{e}_n]_q+\mathbf{L}$$

is a factorization. Using the fact $\mathbf{L} = (\mathbf{L} \cap \mathbf{C}) + \hat{\mathbf{L}}$ is a factorization we conclude that

$$\tilde{\mathbf{L}} = \mathbf{K} + [(1/q)\mathbf{e}_1]_q + \ldots + [(1/q)\mathbf{e}_n]_q + \hat{\mathbf{L}}$$

is a factorization, where $\mathbf{K} = \mathbf{L} \cap \mathbf{C}$. The sets $\tilde{\mathbf{L}}$ and $\hat{\mathbf{L}}$ are abelian groups and $\hat{\mathbf{L}}$ is a subgroup of $\tilde{\mathbf{L}}$ so we can consider the factor group $\tilde{\mathbf{L}}/\hat{\mathbf{L}} = G$. Obviously, G has $(2q)^n$ elements. Denote g_i the coset of $\tilde{\mathbf{L}}/\hat{\mathbf{L}}$ containing the element $(1/q)\mathbf{e}_i$. From the factorization of $\tilde{\mathbf{L}}$ we have the following factorization:

$$G = K + [g_1]_q + \ldots + [g_n]_q.$$

Now we shall prove that $qg_i \notin K - K$ for each i, $1 \leq i \leq n$. Indeed, if $qg_i \in K - K$ had held, we would have had

$$\mathbf{e}_i + \hat{\mathbf{l}}_1 \in (\mathbf{K} + \hat{\mathbf{l}}_2) - (\mathbf{K} + \hat{\mathbf{l}}_3),$$

where $\hat{\mathbf{l}}_1, \hat{\mathbf{l}}_2, \hat{\mathbf{l}}_3 \in \hat{\mathbf{L}}$. Since $\mathbf{L} = \mathbf{K} + \hat{\mathbf{L}}$ and $\hat{\mathbf{L}}$ is a lattice from the assumption $\mathbf{e}_i \in \mathbf{L} - \mathbf{L}$ would follow which is a contradiction.

Note that if $q = rs$, where $r, s \in Z$ and $r > 1, s > 1$, then $[g_1]_{rs} = [g_1]_r + [rg_1]_s$ is a factorization of $[g_1]_{rs}$. If $rg_1 \in K - K$ held, in other words, if $rg_1 = k_1 - k_2$ held, where $k_1, k_2 \in K$, then we would have $rg_1 + k_2 = k_1$ and therefore the element $k_1 \in G$ could be expressed in two distinct forms

$$k_1 + 0 + \ldots + 0, \; k_1 \in K, \; 0 \in [g_1]_q, \ldots, 0 \in [g_n]_q$$

and

$$k_2 + rg_1 + 0 + \ldots + 0, \; k_2 \in K, \; rg_1 \in [g_1]_q, \; 0 \in [g_2]_q, \ldots, \; 0 \in [g_n]_q$$

which would violate the above factorization. We have already proved $srg_1 = qg_1 \notin K - K$. Continuing this way, since q is a power of two, we finally have a factorization

$$G = K + [g_1']_2 + \ldots + [g_m']_2,$$

where $2g_1' \notin K - K$, for each i, $1 \leq i \leq m$. This completes the proof.

Proposition 6. *If there is a counter-example for Keller's conjecture, then there exists a group G which is the internal direct sum of cyclic groups generated by the elements $g_1, \ldots, g_m$ of order four and it has a factorization*

$$G = K' + [g_1]_2 + \ldots + [g_m]_2$$

such that $2g_i \notin K' - K'$ for each i, $1 \leq i \leq m$.

Proof. We shall use the next simple lemma due to S. K. Stein (see [6], p. 545) which enables us to lift a factoring from a group G^* to any group $\bar{G}$ of which G^* is a homomorphic image.

LEMMA. *If $\bar{G}$, G^* are abelian groups, $G^* = A^* + B^*$ is a factorization, φ: $\bar{G} \to G^*$ is a homomorphism from $\bar{G}$ onto G^* and there is a subset A of $\bar{G}$ such that the restriction of φ to A is a bijection between A and A^*, then $\bar{G} = A + B^* \varphi^{-1}$ is a factorization.*

Now, we assume that there is a counter-example for Keller's conjecture, then we may assume that there is an abelian group G and a factorization

$$G = K + [g'_1]_2 + \dots + [g'_m]_2$$

such that $2g'_i \notin K - K$ for each i, $1 \leq i \leq m$ (Proposition 5).

Let G^* be the abelian group generated by the elements $g'_1, \dots, g'_m$ and let $K^* = K \cap G^*$. The above factorization of G is equivalent to that G is disjoint union of the sets $k + [g'_1]_2 + \dots + [g'_m]_2$, where $k \in K$. Since from $k + [g'_1]_2 + \dots + [g'_m]_2 \subset G^*$ it follows that $k \in G^*$, hence

$$G^* = K^* + [g'_1]_2 + \dots + [g'_m]_2$$

is a factorization as well.

Denote $\tilde{\mathbf{L}}$ the lattice spanned by the vectors $(1/2)\mathbf{e}_1, \dots, (1/2)\mathbf{e}_m$ and let φ: $\tilde{\mathbf{L}} \to G^*$ be the mapping given by

$$((z_1/2)\mathbf{e}_1 + \dots + (z_m/2)\mathbf{e}_m)\varphi = z_1 g'_1 + \dots + z_m g'_m, \; z_1, \dots, z_m \in Z.$$

Since G^* is generated by $g'_1, \dots, g'_m$, the reader can readily verify that φ is a homomorphism from $\tilde{\mathbf{L}}$ onto G^*. Obviously, the restriction of φ to the set

$$[(1/2)\mathbf{e}_1]_2 + \dots + [(1/2)\mathbf{e}_m]_2$$

is a bijection between this set and the set

$$[g'_1]_2 + \dots + [g'_m]_2,$$

hence by Stein's lemma there exists a factorization

$$\tilde{\mathbf{L}} = [(1/2)\mathbf{e}_1]_2 + \dots + [(1/2)\mathbf{e}_m]_2 + \mathbf{L},$$

where $\mathbf{L} = K^*\varphi^{-1}$. Thus the system $(\mathcal{C}_O, \mathbf{L})$ is a tiling in $\mathcal{E}^m$. The system $(\mathcal{C}_O, \mathbf{L})$ has not any two cubes having a common $(m-1)$-dimensional face. Indeed, from $\mathbf{e}_i \in \mathbf{L} - \mathbf{L}$ we would have

$$2g'_1 = \mathbf{e}_i\varphi \in \mathbf{L}\varphi - \mathbf{L}\varphi = K^* - K^* \subset K - K$$

which is a contradiction.

Applying the method of Proposition 3 to this system $(\mathcal{C}_O, \mathbf{L})$, we conclude: there exists a counter-example of Keller's conjecture $(\mathcal{C}_O, \mathbf{L}')$ such that $\mathbf{L}'$ is a periodic set with periods $2\mathbf{e}_1, \dots, 2\mathbf{e}_m$ and note $\mathbf{L}' \subset \tilde{\mathbf{L}}$. Thus there exists a factorization

$$\tilde{\mathbf{L}} = [(1/2)\mathbf{e}_1]_2 + \dots + [(1/2)\mathbf{e}_m]_2 + \mathbf{L}'.$$

If $\hat{\mathbf{L}}$ is the lattice spanned by the vectors $2\mathbf{e}_1, \ldots, 2\mathbf{e}_m$ and

$$\mathbf{C} = \{c_1\mathbf{e}_1 + \ldots + c_m\mathbf{e}_m\colon\ 0 \leq c_i < 2,\ 1 \leq i \leq m\},$$

then $\mathbf{L}' = (\mathbf{L}' \cap \mathbf{C}) + \hat{\mathbf{L}}$ is a factorization. Thus finally there exists a factorization

$$\tilde{\mathbf{L}} = K' + [(1/2)\mathbf{e}_1]_2 + \ldots + [(1/2)\mathbf{e}_m]_2 + \hat{\mathbf{L}},$$

where $\mathbf{K}' = \mathbf{L}' \cap \mathbf{C}$.

The factor group $\tilde{\mathbf{L}}/\hat{\mathbf{L}} = G$ is the internal direct sum of cyclic groups, of order four, generated by g_i's, each of which is the coset of the factor group containing $(1/2)\mathbf{e}_i$. The equation

$$G = \mathbf{K}' + [g_1]_2 + \ldots + [g_m]_2$$

is a factorization of G and since $(\mathcal{C}_O, \mathbf{L}')$ is a counter-example for Keller's conjecture, $2g' \notin K' - K'$ for each i, $1 \leq i \leq m$. This completes the proof.

The next proposition presents a contrast to the previous one.

Proposition 7. *Let G be the internal direct sum of cyclic groups of order four generated by elements $g_1, \ldots, g_m$. If there is a factorization*

$$G = K' + [g_1]_2 + \ldots + [g_m]_2$$

such that $2g_i \notin K' - K'$ for each i, $1 \leq i \leq m$, then there is a counter-example for Keller's conjecture.

Proof. Let $\tilde{\mathbf{L}}$ be the lattice spanned by the vectors $(1/2)\mathbf{e}_1, \ldots, (1/2)\mathbf{e}_m$ and let φ: $\tilde{\mathbf{L}} \to G$ the homomorphism from $\tilde{\mathbf{L}}$ onto G given by

$$((z_1/2)\mathbf{e}_1 + \ldots + (z_m/2)\mathbf{e}_m)\varphi = z_1g_1 + \ldots + z_mg_m,\ z_1, \ldots, z_m \in Z.$$

Obviously, the restriction of φ to the set

$$[(1/2)\mathbf{e}_1]_2 + \ldots + [(1/2)\mathbf{e}_m]_2 \subset \tilde{\mathbf{L}}$$

is a bijection between this set and the set

$$[g_1]_2 + \ldots + [g_m]_2 \subset G$$

hence by Stein's lemma there exists a factorization

$$\tilde{\mathbf{L}} = [(1/2)\mathbf{e}_1]_2 + \ldots + [(1/2)\mathbf{e}_m]_2 + \mathbf{L},$$

where $\mathbf{L} = K'\varphi^{-1}$. This means that the system $(\mathcal{C}_O, \mathbf{L})$ is a tiling in $\mathfrak{E}^m$. This system is a counter-example for Keller's conjecture because from $\mathbf{e}_i \in \mathbf{L} - \mathbf{L}$ we would have that $2g_i = \mathbf{e}_i\varphi \in \mathbf{L}\varphi - \mathbf{L}\varphi = K' - K'$.

*

ACKNOWLEDGEMENT. The author would like to thank the referee, Prof. E. MOLNÁR, for his valuable suggestions.

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(Received February 11, 1985)

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