I. Prove carefully from the axioms of ZF set theory that if \( X \) is a set then the collection of all wellorderings of subsets of \( X \) is a set. You should say which axiom(s) are needed at each step in your proof.

Clarification: I mean the collection of those \( S \) such that for some \( Y \subseteq X \), \( S \) is a wellordering of \( Y \).

A wellordering of \( X \) is a subset of \( X^2 \). Each element of \( X^2 \) is in \( P(P(X)) \), so every wellordering is in \( P(P(P(X))) \). So we make three appeals to the axiom of powerset followed by one appeal to comprehension (picking out those elements of \( P(P(P(X))) \) which are wellorderings of \( X \)).

II. Let \( \alpha \) and \( \beta \) be ordinals. Order the cartesian product of \( \alpha \) and \( \beta \), i.e.

\[
\alpha \times \beta := \{(\gamma, \delta) | \gamma < \alpha \land \delta < \beta\}
\]

by \( (\gamma_1, \delta_1) < (\gamma_2, \delta_2) \) if and only if one of the three following conditions holds:

i. \( \gamma_1 = \gamma_2 \land \delta_1 < \delta_2 \)
ii. \( \gamma_1 < \gamma_2 \land \delta_1 = \delta_2 \)
iii. \( \gamma_1 < \gamma_2 \land \delta_1 < \delta_2 \)

Prove that this binary relation is wellfounded.

Let \( X \) be a nonempty subset, let \( \gamma_0 \) be minimal such that \( \exists \delta \ (\gamma_0, \delta) \in X \) and let \( \delta_0 \) be minimal such that \( (\gamma_0, \delta_0) \in X \). Routinely \( (\gamma_0, \delta_0) \) is minimal.

For each of the following values of \( \alpha \) and \( \beta \), determine the least ordinal \( \gamma \) such that there is a function \( F: \alpha \times \beta \rightarrow \gamma \) with the property that

\[
(\gamma_1, \delta_1) < (\gamma_2, \delta_2) \Rightarrow F(\gamma_1, \delta_1) < F(\gamma_2, \delta_2)
\]

Let \( \rho(\alpha, \beta) \) be the sup of the values \( \rho(\gamma', \delta') + 1 \) taken over the set of pairs such that \( (\gamma', \delta') \in R(\gamma, \delta) \).

(a) \( \alpha = \beta = \omega \)

A routine induction shows that \( \rho(m, n) = \omega + m \) for all \( m, n \). So \( \gamma = \omega \).

(b) \( \alpha = \beta = \omega + \omega \)

More routine induction shows that \( \rho(\omega, m) = \omega + m \) for all \( m, \rho(\omega, \omega) = \omega + \omega \), \( \rho(\omega + m, \omega + 1) = \omega + m + 1 \), \( \rho(\omega + m, \omega + n) = \omega + m + n \).

So \( \gamma = \omega + \omega + \omega = \omega.3 \).

III. Recall the definition of ordinal exponentiation: if \( \alpha \) is an ordinal such that \( \alpha > 0 \) then

\[
\alpha^0 = 1 \\
\alpha^{(\beta+1)} = \alpha^\beta \cdot \alpha \\
\alpha^\lambda = \cup \{\alpha^\beta : \beta < \lambda\} \text{ for } \lambda \text{ a limit ordinal.}
\]

Prove that \( \alpha^\beta \) could also have been defined by the following recursion equations:

\[
\alpha^0 = 1, \text{ and for } \beta > 0 \\
\alpha^\beta = \{\alpha^\gamma \cdot \delta + \epsilon : \gamma < \beta, \delta < \alpha, \epsilon < \alpha^\gamma\}
\]

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This is routine, prove by induction on \( \beta \) that both sides are equal.
Using whichever definition you prefer, prove that
\[
\alpha^{(\beta_1 + \beta_2)} = \alpha^{\beta_1} \cdot \alpha^{\beta_2}
\]
This is also a routine induction on \( \beta_2 \).

IV. Let \( X \) be a nonempty set. We make two definitions:
(a) A **filter** on \( X \) is a subset \( F \subseteq \mathcal{P}(X) \) such that

- \( X \) is in \( F \), and \( \emptyset \) is not in \( F \).
- For every \( A \) and \( B \) in \( F \), the intersection of \( A \) and \( B \) is in \( F \).
- For every \( A \) in \( F \), every subset of \( X \) which contains \( A \) (has \( A \) as a subset) is also in \( F \).

(b) An **ultrafilter** on \( X \) is a filter \( U \) on \( X \) with the additional property that for every subset \( A \) of \( X \), exactly one of \( A \) and \( X \setminus A \) is in \( U \).

Prove that \( U \) is an ultrafilter on \( X \) if and only if the following two conditions are satisfied:
- \( U \) is a filter on \( X \).
- There is no filter on \( X \) which properly contains \( U \).

Suppose firstly that \( U \) is a maximal filter. Suppose for a contradiction that \( U \) is not a maximal filter and let \( U \subsetneq F \) for some filter \( F \). Let \( A \in F \setminus U \), then \( A^c \in U \), but this is a contradiction since both \( A \) and \( A^c \) are in \( F \) but their intersection is not in \( F \).

Now suppose that \( U \) is a maximal filter, and let \( A \notin U \). It is easy to see that \( \{ B : A \cup (B \setminus A) \in U \} \) is a filter which contains \( U \). So it is equal to \( U \) and in particular \( A^c \in U \).

Use ZL to show that for every filter on \( X \) there is an ultrafilter \( U \) on \( X \) which contains \( F \).

Partially order the collection of filters by inclusion, check that the union of a chain of filters is a filter, and appeal to ZL.

V. By finding explicit bijections between them, or otherwise, show that all of the following sets have the same cardinality:
(a) \( \mathcal{P}(\omega) \)
(b) \( \mathcal{P}(\omega) \times \mathcal{P}(\omega) \)
(c) The set of all functions from \( \omega \) to \( \omega \)
(d) The set of all permutations of \( \omega \) (that is, functions from \( \omega \) to \( \omega \) which are both injective and surjective)

To see a bijection between the first two, just let \( A \) correspond to \((A_e, A_o)\) where \( A_e = \{ n : 2n \in A \} \) and \( A_o = \{ n : 2n + 1 \in A \} \). Using a bijection between \( \omega \) and \( \omega \times \omega \) we see that both sets are in bijection with \( \mathcal{P}(\omega \times \omega) \).

For the rest, we will make use of the Schroeder-Bernstein theorem. The identity map is an injective map from the set of permutations to the set of functions. Also the identity map is an injective map from the set of functions to \( \mathcal{P}(\omega \times \omega) \). Finally there is an injective map from \( \mathcal{P}(\omega) \) to the set of permutations, in which \( A \) maps to the permutation \( f_A \) which exchanges \( 2n \) and \( 2n + 1 \) for \( n \in A \) while fixing the rest of \( \omega \).