I. Prove carefully from the axioms of ZF set theory that if $X$ is a set then the collection of all wellorderings of subsets of $X$ is a set. You should say which axiom(s) are needed at each step in your proof.

Clarification: I mean the collection of those $S$ such that for some $Y \subseteq X$, $S$ is a wellordering of $Y$.

II. Let $\alpha$ and $\beta$ be ordinals. Order the cartesian product of $\alpha$ and $\beta$, i.e. $\alpha \times \beta := \{(\gamma, \delta) | \gamma < \alpha \land \delta < \beta\}$ by $(\gamma_1, \delta_1) < (\gamma_2, \delta_2)$ if and only if one of the three following conditions holds:

i. $\gamma_1 = \gamma_2 \land \delta_1 < \delta_2$

ii. $\gamma_1 < \gamma_2 \land \delta_1 = \delta_2$

iii. $\gamma_1 < \gamma_2 \land \delta_1 < \delta_2$

Prove that this binary relation is wellfounded.

For each of the following values of $\alpha$ and $\beta$, determine the least ordinal $\gamma$ such that there is a function $F : \alpha \times \beta \to \gamma$ with the property that $$(\gamma_1, \delta_1) < (\gamma_2, \delta_2) \implies F(\gamma_1, \delta_1) < F(\gamma_2, \delta_2)$$

(a) $\alpha = \beta = \omega$

(b) $\alpha = \beta = \omega + \omega$

III. Recall the definition of ordinal exponentiation: if $\alpha$ is an ordinal such that $\alpha > 0$ then

$\alpha^0 = 1$

$\alpha^{(\beta+1)} = \alpha^\beta \cdot \alpha$

$\alpha^\lambda = \cup \{\alpha^\beta : \beta < \lambda\}$ for $\lambda$ a limit ordinal.

Prove that $\alpha^\beta$ could also have been defined by the following recursion equations: $\alpha^0 = 1$, and for $\beta > 0$

$$\alpha^\beta = \{\alpha^\gamma \cdot \delta + \epsilon : \gamma < \beta, \delta < \alpha, \epsilon < \alpha^\gamma\}$$

Using whichever definition you prefer, prove that $\alpha^{(\beta_1+\beta_2)} = \alpha^{\beta_1} \cdot \alpha^{\beta_2}$

IV. Let $X$ be a nonempty set. We make two definitions:

(a) A filter on $X$ is a subset $F \subseteq \mathcal{P}(X)$ such that

- $X$ is in $F$, and $\emptyset$ is not in $F$.
- For every $A$ and $B$ in $F$, the intersection of $A$ and $B$ is in $F$.
- For every $A$ in $F$, every subset of $X$ which contains $A$ (has $A$ as a subset) is also in $F$.

(b) An ultrafilter on $X$ is a filter $U$ on $X$ with the additional property that for every subset $A$ of $X$, exactly one of $A$, and $X \setminus A$ is in $U$.

\textit{Date: Due: 5 March 2009, 17:00.}
Prove that $U$ is an ultrafilter on $X$ if and only if the following two conditions are satisfied:

- $U$ is a filter on $X$.
- There is no filter on $X$ which properly contains $U$.

Use ZL to show that for every filter on $X$ there is an ultrafilter $U$ on $X$ which contains $F$.

V. By finding explicit bijections between them, or otherwise, show that all of the following sets have the same cardinality:

(a) $\mathcal{P}(\omega)$
(b) $\mathcal{P}(\omega) \times \mathcal{P}(\omega)$
(c) The set of all functions from $\omega$ to $\omega$
(d) The set of all permutations of $\omega$ (that is, functions from $\omega$ to $\omega$ which are both injective and surjective)