3.3 We will prove this by induction on $\beta$.

Base case: $\beta = 0$. Then $V_\emptyset = \emptyset$, and

$$\{ \alpha \in V_0 : \alpha \text{ is an ordinal} \} = \emptyset = 0.$$

Successor step: assume that

$$\{ \alpha \in V_\beta : \alpha \text{ is an ordinal} \} = \beta.$$

Let $\alpha \in V_{\beta + 1}$ be an ordinal. Since $\alpha$ is a set of ordinals and $\alpha \subseteq V_\beta$, we have by the induction hypothesis that $\alpha \subseteq \beta$, that is $\alpha \leq \beta$ or equivalently $\alpha < \beta + 1$.

Conversely let $\alpha \leq \beta$. There are two cases:

(a) $\alpha < \beta$. In this case by the induction hypothesis $\alpha \in V_\beta$. Since $V_\beta \subseteq V_{\beta + 1}$ it follows that $\alpha \in V_{\beta + 1}$.

(b) $\alpha = \beta$. Since $\beta \subseteq V_\beta$, it follows that $\alpha \in V_{\beta + 1}$.

We showed that

$$\{ \alpha \in V_{\beta + 1} : \alpha \text{ is an ordinal} \} = \beta + 1.$$

Limit step: assume that $\lambda$ is a limit ordinal and $\{ \alpha \in V_\beta : \alpha \text{ is an ordinal} \} = \beta$ for all $\beta < \lambda$. Then $V_\lambda = \bigcup_{\beta < \lambda} V_\beta$, and so

$$\{ \alpha \in V_\lambda : \alpha \text{ is an ordinal} \} = \bigcup_{\beta < \lambda} \beta = \lambda.$$

This proof used the “case by case” recursive definition of $V_\beta$, I could have given a similar argument using the slick definition $V_\beta = \bigcup_{\alpha < \beta} P(V_\alpha)$.

3.4 I’ll prove these using the recursive definition

$$\alpha + \beta = \alpha \cup \{ \alpha + \gamma : \gamma < \beta \}$$

One can certainly also prove them by exhibiting isomorphisms between the wellorderings which define the various ordinal sums.

(a) $\alpha + 0 = \alpha \cup \{ \alpha + \gamma : \gamma < 0 \} = \alpha$. We prove by induction on $\alpha$ that $0 + \alpha = \alpha$, this is easy: assuming that $0 + \gamma = \gamma$ for $\gamma < \alpha$,

$$0 + \alpha = 0 \cup \{ 0 + \gamma : \gamma < \alpha \} = \{ \gamma : \gamma < \alpha \} = \alpha.$$

(b) We prove this by induction on $\beta$. Assume that $\gamma \leq \alpha + \gamma$ for $\gamma < \beta$, and let $\zeta < \beta$.

Then $\zeta \leq \alpha + \zeta$ by the induction hypothesis, $\alpha + \zeta < \alpha + \beta$ by the recursive definition of $+$, so $\zeta < \alpha + \beta$. It follows that $\beta \leq \alpha + \beta$.

If you find this opaque: keep in mind that for $\alpha, \beta$ which are ordinals $\alpha < \beta \iff \alpha \subseteq \beta$.

(c) It is immediate from the recursive definition of $+$ that for every $\beta < \gamma, \alpha + \beta < \alpha + \gamma$.

(d) Fix $\alpha \leq \beta$, we establish the result by induction on $\gamma$. Suppose that $\alpha + \delta \leq \beta + \delta$ for all $\delta < \gamma$. Let $\zeta < \alpha + \gamma$. There are two cases:

(i) $\zeta < \alpha$, in which case $\zeta < \beta$ and so $\zeta < \beta + \gamma$.

(ii) $\zeta = \alpha + \delta$ for $\delta < \gamma$, in which case $\zeta \leq \beta + \delta$ by the induction hypothesis, $\beta + \delta < \beta + \gamma$ by the recursive definition of $+$, and hence $\zeta < \beta + \gamma$.

3.6 Again we can do it by construction of explicit isomorphisms, but I use the recursion equation

$$\alpha.\beta = \{ \alpha.\gamma + \delta : \gamma < \beta, \delta < \alpha \}$$

(a) Directly from the definition $\alpha.0$ and $0.\alpha$ are empty.
(b) $\alpha.1 = \{\alpha.0 + \delta : \delta < \alpha\} = \{\delta : \delta < \alpha\} = \alpha$.

By induction on $\alpha$,

$$1.\alpha = \{1.\gamma + 0 : \gamma < \alpha\} = \{\gamma : \gamma < \alpha\} = \alpha.$$  

(c) Directly from the definition, if $0 < \alpha$ and $\beta < \gamma$ then $\alpha.\beta = \alpha.\beta + 0 < \alpha.\gamma$.

(d) We prove it by induction on $\gamma$ for fixed $\alpha \leq \beta$. Suppose that $\alpha.\zeta \leq \beta.\zeta$ for all $\zeta < \gamma$. Let $\eta < \alpha.\gamma$, so that $\eta = \alpha.\zeta + \delta$ for some $\delta < \alpha$. Now $\alpha.\zeta + \delta \leq \beta.\zeta + \delta$ by a previous exercise, and $\beta.\zeta + \delta < \beta.\gamma$ by definition. So $\eta < \beta.\gamma$.

(e) We did this one in class.

**The ordinal $\omega_1$**

Let $\omega_1$ be the collection of all countable ordinals. We will show that this is a transitive set of ordinals.

Transitive: let $\alpha \in \beta$ where $\beta$ is a countable ordinal. Then $\alpha \subseteq \beta$, so $\alpha$ is also countable.

Set: consider the operation which maps each wellordering of $\omega$ to its order type, and use the axiom of replacement.

A transitive set of ordinals is an ordinal, so $\omega_1$ is an ordinal. No ordinal can be a member of itself, so $\omega_1$ is uncountable. Also for every ordinal $\alpha$, $\alpha$ is countable iff $\alpha \in \omega_1$ iff $\alpha < \omega_1$, so $\omega_1$ is the least uncountable ordinal.

$\alpha + \beta = \beta$? Fix $\alpha$ and let $\beta = \alpha.\omega$. Then

$$\alpha + \beta = \alpha.1 + \alpha.\omega = \alpha.(1 + \omega) = \alpha.\omega = \beta.$$  

Remark: any ordinal $\beta \geq \alpha.\omega$ also works.