(1) Proof 1: Let $A$ be a nonempty subset of $\omega^2$, let $a = \min \{x : (x, y) \in A\}$ and $b = \min \{y : (a, y) \in A\}$. Then $(a, b)$ is minimal in $A$.

Proof 2: Suppose for contradiction that $(a_n, b_n)$ for $n \in \omega$ form an infinite decreasing sequence. Then $a_{n+1} \leq a_n$ for all $n$ so $a_n$ is eventually constant. Say $a_n = A$ for all $n \geq N$. But then $b_{n+1} < b_n$ for $n \geq N$, contradiction.

(2) The union of $C$ is clearly a subset of $X^2$, thus it is a binary relation on $X$. We call this relation $R$ and check that $R$ has the needed properties.

(a) Let $a \in X$. Since $C$ is nonempty there is $S \in C$, and since $S$ is a partial ordering $S$ is reflexive. Hence $(a, a) \in S$, so $(a, a) \in R$.

(b) Let $(a, b) \in R$ and $(b, c) \in R$. By definition there are $S_1, S_2$ in $C$ with $(a, b) \in S_1$ and $(b, c) \in S_2$. Since $C$ is a chain either $S_1 \subseteq S_2$ or $S_2 \subseteq S_1$, hence for some $i \in \{1, 2\}$ we have $(a, b), (b, c)$ both in $S_i$. Now $S_i$ is transitive so $(a, c) \in S_i$ and thus $(a, c) \in R$.

(c) Let $(a, b)$ and $(b, a)$ both be in $R$. As in the last part there is $S \in C$ with $(a, b)$ and $(b, a)$ both in $S$. As $S$ is a partial ordering $a = b$.

(3) We show that a partial ordering $R$ is linear iff it is maximal.

Linear implies maximal: let $R$ be linear and let $S$ be a partial ordering with $R \subseteq S$. Let $aSb$, we will show that $aRb$.

As $R$ is linear we know that either $aRb$ or $bRa$. If $aRb$ we are done. If $bRa$ then $bSa$ since $R \subseteq S$. But then $aSb$ and $bSa$, so $a = b$, and then $aRb$ because $R$ is reflexive.

Maximal implies linear: We show that if $R$ is not linear then it is not maximal. Suppose that $R$ is not linear and fix $a, b$ such that neither $aRb$ nor $bRa$ holds. We will define a larger partial ordering in which $a$ is related to $b$.

Define $S$ as follows: $xSy$ iff either $xRy$ or $xRa$ and $bRy$. We claim this is a partial order:

(a) For all $x$, $xRa$ and so $xSx$.

(b) Let $xSy$ and $ySz$. There are (a priori) four possibilities:

(i) $xRy$ and $yRz$. In this case $xRz$ by transitivity of $R$, so $xSx$.

(ii) $xRy$, $yRa$ and $bRz$. Then $xRa$ by transitivity of $R$, so $xSx$.

(iii) $xRa$, $bRy$ and $yRz$. Then $bRz$, so $xSx$.

(iv) $xRa$, $bRy$, $yRa$, $bRz$. This is actually impossible because $bRy$ and $yRa$ would imply that $bRa$, which is not the case.

(c) Let $xSy$ and $ySz$. We do the same case analysis as above:

(i) $xRy$ and $yRx$. In this case $xRy$ since $R$ is a partial ordering.

(ii) $xRy$, $yRa$ and $bRx$. Then $bRz$ by transitivity of $R$, which is impossible.

(iii) $xRa$, $bRy$ and $yRx$. Then $bRa$, which is impossible

(iv) $xRa$, $bRy$, $yRa$, $bRz$. This implies $bRa$, which is impossible.

So $x = y$.

We showed that $S$ is a strictly larger partial ordering than $R$.

(4) We take each claim in turn:

(a) It is transitive. let $fRgRb$. Find $m_1$ such that $f(n) < g(n)$ for $n > m_1$ and $m_2$ such that $g(n) < h(n)$ for $n > m_2$. If $m = \max(m_1, m_2)$ then $f(n) < h(n)$ for $n > m$.

(b) It is not well-founded. Define for each $i$ a function $f_i$ such that $f_i(m) = \max(0, m - i)$. It is easy to see that $f_{i+1}Rf_i$ for all $i$.

(c) It is true that every countable set has this property. Let us enumerate the set $Y$ as $f_0, f_1, \ldots$ and define $h(n) = \sum_{i=0}^{\infty} f_i(n) + 1$. Then $h(n) > f_i(n)$ for $n \geq i$. 

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