2.10 (a) This is routine: $0 = \emptyset$ and so $P(0) = \{0\}$, a set with $1 = 2^0$ elements.  

Induction step: assume that $P(n)$ has $2^n$ elements. We know that $n+1 = n \cup \{n\}$, so that every subset of $n+1$ is of the form $A$ or $A \cup \{n\}$ for some subset $A$ of $n$; hence $P(n+1)$ has $2 \times 2^n = 2^{n+1}$ many elements.  

Note: You do not really need induction to prove this. Also the proof given here is quite informal, because we have not yet learned how to write a formal inductive proof in set theory nor had a careful discussion of the cardinality (size) of a set.  

(b) Easiest to build it up in steps: note that I am listing the elements in increasing order of size, and am using abbreviations wherever possible.  

(i) $V_1 = \{0\} = 1$.  

(ii) $V_2 = \{0, \{0\}\} = \{0, 1\} = 2$.  

(iii) $V_3 = \{0, 1, \{1\}\}$.  

(iv) $V_4 = \{0, 1, \{1\}, 2, 0, \{0, 1\}, \{0, 2\}, \{1, \{1\}\}, \{1, 2\}, \{\{1\}, 2\}, \{0, 1, \{1\}, 2\}, 1, \{1, 2\}, V_3\}$.  

(c) $V_{n+1} = P(V_n)$, so by the first part $|V_{n+1}| = 2^{|V_n|}$. So by induction $|V_n|$ is the number obtained by iterating the map $a \mapsto 2^a$ on argument zero $n$ many times.  

2.3 Note that we can write the condition for transitivity as $x \subseteq P(x)$ or $\bigcup \Delta x \subseteq x$.  

(a) We proceed by induction on $n$.  

Base case: $V_0 = \emptyset$, so there is nothing to check.  

Induction step: Assume that $V_n$ is transitive. Let $A \in V_{n+1}$, so that $A \subseteq V_n$ by definition. Then for any $x$  

$$x \in A \implies x \in V_n \implies x \subseteq V_n \implies x \in V_{n+1},$$  

and we are done.  

(b) Again we use induction on $n$.  

Base case: $V_0 = \emptyset \subseteq V_1$.  

Induction step: Assume that $V_n \subseteq V_{n+1}$. Let $A \in V_{n+1}$, then $A \subseteq V_n \subseteq V_{n+1}$, so that $A \subseteq V_{n+1}$ and hence $A \in V_{n+2}$.  

(c) Once more we use induction.  

Base case: $V_0 \cap \omega = 0$.  

Induction step: Assume that $V_n \cap \omega = n$. It follows that $n \subseteq V_n$, so that $n \in V_{n+1}$. Also $n \subseteq V_n \subseteq V_{n+1}$, and so $n+1 = n \cup \{n\} \subseteq V_{n+1}$. Also $n+1 = \{0, \ldots, n\} \subseteq \omega$ and hence $n+1 \subseteq V_{n+1} \cap \omega$.  

For the converse inclusion let $m \in \omega \cap V_{n+1}$. Then $m \subseteq \omega$ (because $\omega$ is a transitive set) and $m \subseteq V_n$, so that $m \subseteq n$. It follows that $m \leq n$ and so $m \in n+1$.  

2.10 (a) This is routine: $x\Delta x$ is empty, $x\Delta y = y\Delta x$ so reflexive and symmetric properties are immediate. As for transitivity note that $x\Delta z = (x\Delta y)\Delta(y\Delta z)$.  

(b) Fix $x$ and let $x_n = x\Delta\{n\}$. This generates an infinite family of distinct sets each equivalent to $x$.  

(c) To get an infinite family: let $A_n = \{2^m3^n : m, n \in \omega\}$, these are infinitely many inequivalent sets.  

First proof that there are uncountably many classes: let $X_n \subseteq \omega$ for $n \in \omega$. Define a set $Y = \{2^m3^n : m, n \in \omega, 2^m3^n \not\in X_n\}$. Now $Y\Delta X_n \supseteq A_n$ so $Y$ is not equivalent to $X_n$.  

Second proof: each class is countable (by last Q on HW). There are uncountably many subsets of $\omega$.  


Let $p_1, p_2, \ldots$ enumerate the set of primes in increasing order. Given a finite subset $a \subseteq \omega$, enumerate $a$ as $a_1 < \ldots < a_t$ and then code it by the natural number $p_{a_1+1} \cdots p_{a_t+1}$. The map from finite sets to codes is injective so there are only countably many finite subsets of $\omega$.

Corollary: each equivalence class in the last question is countable, because the class of $x$ is the set of $x \Delta a$ for a finite.