This final is due by noon on Mon 11 May. Please submit electronically as a PDF file emailed to me. You may work on the exam during any continuous 24 hour period before the due date. You may not collaborate, but may consult any printed or online source (please cite what you use). Please contact me by email if you have any questions or would like to arrange a meeting.

You should attempt exactly five questions. All questions carry equal weight.

1. (a) Show that every open set in \( \mathbb{R} \) is a countable union of closed sets.
   (b) Give an explicit example of a countable union of closed sets which is neither open nor closed.
   (c) Prove that if \( X \subseteq \mathbb{R} \) is a countable union of closed sets, then either \( X \) is countable or \(|X| = 2^{\aleph_0}\).

2. (a) Prove that \( < \) is a wellordering of \( S \).
   (b) What is the order type of \( (S, <) \)?
   (c) What is the order type of the subset of \( S \) consisting of strictly decreasing finite sequences?

3. Consider the following property of an ordinal \( \beta \): for all \( X, Y \subseteq \beta \) such that \( \beta = X \cup Y \), at least one of \( X, Y \) has order type \( \beta \).
   (a) Prove that for every \( \alpha \), the ordinal \( \omega^\alpha \) (I mean ordinal exponentiation here) has this property.
   (b) Prove that no other ordinals have this property.

4. Recall the infinite Ramsey theorem: if \( f : [\omega]^n \to k \) for \( n, k \) finite there is an infinite \( H \subseteq \omega \) such that \( f \restriction [H]^n \) is constant.
   (a) Let \( f : \omega \to \omega \).
      Prove that there is an infinite \( H \subseteq \omega \) such that \( f \restriction H \) is either constant or 1-1. Hint: use Ramsey for pairs.
   (b) Let \( f : [\omega]^2 \to \omega \). Prove that there is an infinite set \( H \subseteq \omega \) such that one of the following holds:
      (i) \( f \restriction [H]^2 \) is constant.
      (ii) \( f \restriction [H]^2 \) is 1-1.
      (iii) There is a 1-1 function \( g : H \to \omega \) such that \( f(a, b) = g(a) \) for all \( a, b \in H \).
      (iv) There is a 1-1 function \( h : H \to \omega \) such that \( f(a, b) = h(b) \) for all \( a, b \in H \).

5. Let \( \pi \) be a bijection from \( V \) to \( V \). Define a relation \( \in' \) on \( V \) by
   \[ x \in' y \iff x \in \pi(y). \]
   For each formula \( \phi \) in the language of set theory, let \( \phi' \) be the formula obtained by replacing \( \in \) by \( \in' \).
   (a) Show that if \( \phi \) is the axiom of foundation then there is a permutation \( \pi \) such that \( \phi' \) is false.
   (b) Choose three other axioms, and show that their primed versions are true for every \( \pi \).

6. Show that for any set \( X \), every well-founded partial ordering of \( X \) can be extended to a wellordering of \( X \).

7. Fill in the following sketch of an alternative proof that every uncountable closed set of reals is the union of a perfect closed set and a countable set.
   Let \( X \) be an uncountable closed set of reals, and let \( Y \) be the set of \( x \in X \) such that \( X \cap (x - \epsilon, x + \epsilon) \) is uncountable for all \( \epsilon > 0 \).
(a) $X \setminus Y$ is countable.
(b) $Y$ is not empty.
(c) $Y$ is closed.
(d) $Y$ is perfect.

(8) Let $n$ be a natural number.
(a) Prove that for every $f : \omega \to \omega_{n+1}$ there is $\alpha < \omega_{n+1}$ such that $\text{rge}(f) \subseteq \alpha$.
(b) Prove that $\aleph_{n+1}^{\aleph_0} = \max\{\aleph_{n+1}, 2^{\aleph_0}\}$. 