(1) Read sections 1-4 of chapter I in Kunen.

Easy.

(2) Define a binary relation $<_*$ on $\mathbb{N}^\mathbb{N}$ (the set of all functions from $\mathbb{N}$ to $\mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers) as follows: $f <_* g$ if and only if there is a natural number $M$ such that $f(i) < g(i)$ for all $i > M$. This is usually called the “eventual domination” relation.

Show that if $X$ is a countable subset of $\mathbb{N}^\mathbb{N}$ then there is $g$ in $\mathbb{N}^\mathbb{N}$ such that $f <_* g$ for all $f \in X$.

Let $F : \mathbb{N} \to X$ be surjective and let $g(j) = \sum_{i \leq j} F(i)(j) + 1$.

(3) Two infinite subsets of $\mathbb{N}$ are said to be “almost disjoint” when they have finite intersection. Show that there exists a collection of size $2^{\aleph_0}$ of infinite subsets of $\mathbb{N}$ such that any two are almost disjoint.

Note that it suffices to produce such a family of subsets of any countably infinite set (then we can copy it over to $\mathbb{N}$ via any bijection). Let $\text{Seq}$ be the set of finite sequences of integers, this is countably infinite. For any $f : \mathbb{N} \to \mathbb{N}$ let $X_f \subseteq \text{Seq}$ be the set

$$\{(f(0)), (f(0), f(1)), (f(0), f(1), f(2)), \ldots\}$$

of finite initial segments of $f$. Clearly if $f \neq g$ then $X_f \cap X_g$ is finite.

(4) If $f$ and $g$ are in $\mathbb{N}^\mathbb{N}$ define $d(f, g)$ to be 0 if $f = g$, and to be $2^{-x}$ for $x$ minimal with $f(x) \neq g(x)$ otherwise. Show that this makes $\mathbb{N}^\mathbb{N}$ into a complete metric space.

It is routine to check that that $d$ is a metric. For completeness let $(f_i)$ be a Cauchy sequence. Observe that by Cauchy-ness and the defn of the metric, for every $m$ there is $i$ such that $f_j(m)$ is constant for $j \geq i$; define $f(m)$ to be the eventual value of $f_i(m)$ and check that $(f_i)$ converges to $f$.

Show further that as a topological space this space is homeomorphic to the product of countably many copies of $\mathbb{N}$ each equipped with the discrete topology.

For each finite sequence $s$ let $N_s$ be the set of function which have $s$ as an initial segment. Then it is routine to check that both these topologies have $\{N_s : s \in \text{Seq}\}$ as a basis so they coincide.