MATRIX THEORY AUTUMN 2011: MIDTERM (TAKE HOME PART)

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This exam is due by noon on Friday 21 October. It should either be delivered to my office (Wean Hall 7101) or by email (PDF and JPEG are the only acceptable formats). Your answers should be typeset or written in black on alternate lines. You may ask me for clarification of a question but may not consult anyone else. You may consult any written or online source, but you must cite any source other than course notes or previous HW sets and solutions. All questions carry equal weight.

All matrices and vectors are assumed to have real entries.

(1) Let $n > 0$. Prove that for any $n \times n$ matrix A, the following statements are equivalent:
   (a) $AB = BA$ for all $n \times n$ matrices B.
   (b) $AB = BA$ for all invertible $n \times n$ matrices B.

   Hint: You may find it helpful to identify the class of matrices A which satisfy the first property.

(2) Let $n > 0$ and let A be an $n \times n$ matrix. For all $t \geq 0$, let N_t be the nullspace of A^t, where by convention $A^0 = 1_{n \times n}$.

   Prove that:
   (a) $N_t \subseteq N_{t+1}$ for all t.
   (b) The dimension of N_t (the nullity of A^t) is eventually constant, that is there is a number d such that $dim(N_t) = d$ for all sufficiently large t.
   (c) If T is the least t such that $dim(N_t) = d$, then $T \leq d$.

   Prove that for all integers $n > 0$, d and T such that $0 < T \leq d \leq n$ there is a matrix A such that the nullity of A^t is d for all $t \geq T$, and T is the least t such that the nullity of A^t is d.

(3) Let $0 < m < n$. Let A be $n \times m$ and let B be $m \times n$. Prove that AB is not invertible. Is it true in general that BA is not invertible?

(4) For the purposes of the following question we will identify the real number r with the 1×1 matrix whose only entry is r. In particular when we write "$M \geq 0$" when M is 1×1 we mean that the entry of M is non-negative.

   Let $m > 1$ and let X be a $1 \times m$ matrix, that is a row vector of length m. Prove that:
   (a) $XX^T \geq 0$.
   (b) XX^T has rank one if $X \neq 0$.
   (c) $X^T X$ is symmetric.
   (d) $A(X^T X)A^T \geq 0$ for all row vectors A of length m.