MATRIX THEORY AUTUMN 2011: HOMEWORK 1 SOLUTIONS

JAMES CUMMINGS

Homework due at start of class on Wed 7 September.

1. Sets and logic

(1) Using truth tables or otherwise, prove that:

(a) "A implies B" is logically equivalent to "not-A or B".

A couple of proofs:

Proof 1: To show that two propositions are equivalent it is enough to show that they are true in exactly the same circumstances, or that they are false in exactly the same circumstances.

"A implies B" is false $\iff$ A is true and B is false $\iff$ "not A" is false and B is false $\iff$ "not A or B" is false.

Proof 2: Truth table proof.

A	B	$A \implies B$	$\neg A$	B	$\neg A \vee B$
F	F	T	T	F	T
F	T	T	T	T	T
T	F	F	F	F	F
T	T	T	F	T	T

(b) "not-(A or B)" is logically equivalent to "not-A and not-B".

Proof 1: "not-(A or B)" is true $\iff$ "A or B" is false $\iff$ A is false and B is false $\iff$ "not A" is true and "not B" is true $\iff$ "not-A and not-B" is true.

Proof 2: Truth table proof.

A	B	$A \vee B$	$\neg(A \vee B)$	$\neg A$	$\neg B$	$\neg A \wedge \neg B$
F	F	F	T	T	T	T
F	T	T	F	T	F	F
T	F	T	F	F	T	F
T	T	T	F	F	F	F

(c) "not-(A implies B)" is logically equivalent to "A and not-B".

We could do it by either of the methods used above, but it's quicker to combine the previous two results.

"not-(A implies B)" $\iff$ "not-(not-A or B)" $\iff$ "not-(not-A) and not-B" $\iff$ "A and not-B".

(2) Recall that the basic operations on sets are intersection, union and set difference. Let X be a set, let $\phi(x)$ and $\psi(x)$ be properties that elements $x \in X$ may have. Let $A = \{x \in X : \phi(x)\}$ and let $B = \{x \in X :$

$\psi(x)\}$. Describe (with proof) each of the following sets in terms of the basic operations:

(a) $\{x \in X : \text{not } \phi(x)\}$.
We claim this is $X \setminus A$, which we prove by showing that the two sets have the same members. $y \in X \setminus A \iff y \in X$ and $y \notin A \iff y \in X$ and not $\phi(x) \iff y \in \{x \in X : \text{not } \phi(x)\}$.

(b) $\{x \in X : \phi(x) \text{ or } \psi(x)\}$.
$A \cup B$.
$y \in A \cup B \iff y \in A$ or $y \in B \iff (y \in X$ and $\phi(y))$ or $(y \in X$ and $\psi(y)) \iff y \in X$ and $(\phi(x)$ or $\psi(x)) \iff y \in \{x \in X : \phi(x) \text{ or } \psi(x)\}$.

(c) $\{x \in X : \phi(x) \text{ implies } \psi(x)\}$.
$(X \setminus A) \cup B$.
We use an earlier problem. $y \in (X \setminus A) \cup B \iff y \in X$ and $(y \notin A$ or $y \in B) \iff y \in X$ and $(\phi(x)$ implies $\psi(x))$

2. Linear equations

(1) Consider linear systems in two variables x, y of the general form

$$\begin{aligned} x + 2y &= 1 \\ ax + y &= b \end{aligned}$$

where a, b are constants.

For which values of a, b does this system have

(a) Exactly one solution?
(b) No solutions?
(c) Infinitely many solutions?

Before we start, a note on the logic of equation solving. Suppose we have a set $E = \{e_1, \dots e_m\}$ of equations. By algebraic manipulation we may deduce some new equations $F = \{e_{m+1}, \dots e_n\}$. Assuming we did not make an algebra mistake every solution of E is a solution of F, but in general F may have more solutions. For example if we square the equation $x = 2$ we get $x^2 = 4$. For the purposes of this question we want to be careful about this issue.

We start by eliminating y, which we do by subtracting twice equation two from equation one. Key point: the new system below is equivalent to the system we started with, since we can recover the old system by adding twice the second equation of the new system to the first equation of the new system.

$$\begin{aligned} (1 - 2a)x &= 1 - 2b \\ ax + y &= b \end{aligned}$$

Claim: if $2a \neq 1$ there is a unique solution.

Proof: Since $2a - 1$ we can divide by it to determine the value of x, then use $y = b - ax$ to find y.

So suppose now that $2a = 1$. The first equation is only solvable if $b = 1/2$, and in this case it becomes the identity $0 = 0$. So if $b \neq 1/2$ there

is no solution, while if $b = 1/2$ then any choice of (x, y) such that $y = b - ax$ works.

3. Matrices and vectors

(1) Linear transformations.

(a) Prove that if T is a linear transformation then there is exactly one matrix A as above.
By definition there is at least one such A. To show it is unique suppose that B is any matrix with the required property. By inspecting $T(1, 0)$ and $T(0, 1)$ we can deduce the entries in the first and second columns of B, so there is only one possibility for B.

(b) Prove that a rotation by ϕ radians about the origin $(0, 0)$ is a linear transformation and describe the corresponding matrix.
We compute the image of the point (x, y). This point is reached by starting at the origin, going x units along the x-axis and then y units along the y-axis. So if T is the rotation map then $T(x, y)$ is obtained by going x units along the line which makes an angle ϕ with the x-axis (which by trigonometry brings us to $(x\cos(\phi), x\sin(\phi))$), and then y units along the line which makes an angle $\phi + \pi/2$ with the x-axis (which brings us to $(x\cos(\phi) - y\sin(\phi), x\sin(\phi) + y\cos(\phi))$).
A little thought shows that

$$\begin{pmatrix} x\cos(\phi) - y\sin(\phi) \\ x\sin(\phi) + y\cos(\phi) \end{pmatrix} = \begin{pmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}$$

(c) Prove that the translation $(x, y) \mapsto (x + 1, y + 1)$ is *not* a linear transformation.
It is easy to see that any linear transformation fixes $(0, 0)$.

(d) Consider the line L with equation $y = ax$. Is reflection in L a linear transformation? If yes find the corresponding matrix.
We compute the image of the point $P = (x, y)$.
If $a = 0$ then the line is $y = 0$ and the image is just $(x, -y)$.
So suppose that $a \neq 0$. Let $Q = (x_0, y_0)$ be the point on $y = ax$ closest to (x, y). If $y \neq ax$ then the line PQ has slope $-a^{-1}$ so we have equations $y_0 = ax_0$ and $y - y_0 = -a^{-1}(x - x_0)$. Some tedious algebra gives $x_0 = b^{-1}(y + xa^{-1})$ and $y_0 = ax_0$ where $b = a + a^{-1}$. This formula also works when $y = ax$.
Now easily the reflection of (x, y) has coordinates $(2x_0 - x, 2y_0 - y)$ and

$$2x_0 - x = 2b^{-1}y + (2b^{-1}a^{-1} - 1)x, 2y_0 - y = (2ab^{-1} - 1)y + 2b^{-1}x.$$

So the final matrix is

$$\begin{pmatrix} 2b^{-1}a^{-1} - 1 & 2b^{-1} \\ 2b^{-1} & 2ab^{-1} - 1 \end{pmatrix}$$

(e) Let A_ϕ be the matrix for rotation by ϕ. Prove that $A_\phi A_\psi = A_{\phi+\psi}$.
Routine calculation.

(2) Let X be the set of 2×2 real matrices. Prove or give a counterexample to each of the following statements:

(a) $A + B = B + A$ for all $A, B \in X$.
True: the (i, j) entry is $a_{ij} + b_{ij} = b_{ij} + a_{ij}$.
(b) $AB = BA$ for all $A, B \in X$.
False: let
$$A = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$
and
$$B = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$$
(c) $A^2 = 0$ implies $A = 0$ for all $A \in X$.
False: look at B from the last part, $B^2 = 0$ but $B \neq 0$.
(d) $AA^2 = A^2A$ for all $A \in X$.
True by associativity of matrix multiplication.

4. Maple

This section is *not for credit*, *you may collaborate* and *you need not hand anything in.* But you still should do it because probably none of this will be true of the Maple sections in later homeworks.

(a) Either install Maple (CMU has a license) on your personal computer or find a public workstation which has it installed.
(b) Start Maple.
(c) Open up a new Maple worksheet.
(d) Load the "LinearAlgebra" package with the command `with(LinearAlgebra);`
(e) Locate the online help for the LinearAlgebra package.
(f) Create matrices
$$A = \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}, B = \begin{pmatrix} 3 & 7 \\ 6 & 8 \end{pmatrix}.$$
(g) Use Maple to compute $A + B, AB, BA, A^{100}$.
(h) Export your results to a PDF file.