MATH STUDIES ALGEBRA SPRING 2018: HOMEWORK 9

JC

This homework is due by class time on Monday 23 April. It must be typeset (preferably in LaTeX) and submitted as a PDF file on the Canvas site, with a filename of the form

`andrewID_alg_homeworknumber.pdf`

For each minute that it is late, the grade will be reduced by 10 percent.

(1) Let $k = \mathbb{Z}/p\mathbb{Z}$, let $k[t]$ be the ring of polynomials in one variable t, and let $l = k(t)$ be the field of fractions of $k[t]$. Let $f = x^p - t \in l[x]$.
  (a) Prove that f is irreducible in $l[x]$.
  (b) Prove that the splitting field of f over l has the form $l(\alpha)$ where $\alpha^p = t$, and describe how f splits in $l(\alpha)[x]$.
  (c) Prove that f is not separable.

(2) Let k be an extension field of l, and say that a set $A \subseteq l$ is *algebraically independent over* k if for every distinct $a_1, \ldots a_n$ and every nonzero $f \in k[x_1, \ldots x_n]$ we have $f(a_1, \ldots a_n) = 0$.
  (a) Prove that there is a maximal algebraically independent set $A \subseteq l$.
  (b) Prove that if A is a maximal algebraically independent subset of l then every element of l is algebraic over $k(A)$.
  (c) One can show (using properties from HW8 Q1, similar to proofs in linear algebra) that any two maximal algebraically independent sets have the same size. We define the *transcendence degree of* l *over* k to be the size of a maximal algebraically independent set, and refer to such sets as *transcendence bases for* l *over* k. Prove that the transcendence degree of $\mathbb{R}$ over $\mathbb{Q}$ is uncountable.

(3) Prove that if l is an algebraic extension of k and every monic irreducible $f \in k[x]$ has at least one root in l, then l is algebraically closed. Such extensions of k are called *algebraic closures of* k.

(4) Fill in the details of the following proof that every field k has an algebraic closure.
  (a) To every monic irreducible $f \in k[x]$ associate a variable symbol t_f, let R be the ring of polynomials in the variables t_f with coefficients from k, and let J be the ideal of R generated by all the polynomials of the form $f(t_f)$. Then $J \neq R$.
  (b) There is a maximal ideal M of R with $M \supseteq J$.
  (c) The field $l = R/M$ contains an isomorphic copy of k, given by $\{a + M : a \in k\}$.
  (d) Identifing $a \in k$ with $a + M \in l$, l is an algebraic closure of k.

(5) Let $\alpha = 2^{1/3}$ and $\beta = \exp(2\pi i/3)$, and let $E = \mathbb{Q}(\alpha, \beta)$. As we saw in class, E is a splitting field for $x^3 - 2$ over $\mathbb{Q}$ and $[E : \mathbb{Q}] = 6$.
  (a) Find a basis for E as a VS over $\mathbb{Q}$.

(b) Describe $Aut(E/\mathbb{Q})$.
(c) For each subgroup H of $Aut(E/\mathbb{Q})$, compute $Fix(H) = \{a \in E : \forall \sigma \in H\ \sigma(a) = a\}$.