MATH STUDIES ALGEBRA SPRING 2018: HOMEWORK 8

JC

This homework is due by class time on Monday 16 April. It must be typeset (preferably in LaTeX) and submitted as a PDF file on the Canvas site, with a filename of the form

`andrewID_alg_homeworknumber.pdf`

For each minute that it is late, the grade will be reduced by 10 percent.

(1) Let k be a subfield of l. We write $P(l)$ for the set of subsets of l, and define a map acl from $P(l)$ to $P(l)$ as follows: $acl(X)$ is the set of $b \in l$ such that b is algebraic over $k(X)$. Prove that:
  (a) $X \subseteq acl(X)$.
  (b) If $X \subseteq Y$ then $acl(X) \subseteq acl(Y)$.
  (c) $acl(acl(X)) = acl(X)$.
  (d) If $b \in acl(X)$ then there is finite $x \subseteq X$ such that $b \in acl(x)$.
  (e) If $b \in acl(X \cup \{a\}) \setminus acl(X)$, then $a \in acl(X \cup \{b\})$.
  What do these properties remind you of?

(2) Let k be a subfield of l and let $a, b \in l \setminus k$ be such that $[k(a) : k]$ and $[k(b) : k]$ are coprime. Prove that $m_a^k = m_a^{k(b)}$.

(3) Let p be a prime and let k be a field of characteristic p. Define $\phi : k \to k$ by $\phi : a \mapsto a^p$. Prove that:
  (a) The map ϕ is a monomorphism from k to k.
  (b) If k is finite then ϕ is an automorphism of k.
  (c) If k is finite and $|k| = p^n$ then ϕ is an element of order n in the group $Aut(k)$.

(4) Let $k = \mathbb{Z}/2\mathbb{Z}$. Find an irreducible $f \in k[x]$ of degree three. Now let $l = k[x]/(f)$, let $\alpha = x + (f)$ and identify $a \in k$ with $a + (f) \in l$.
  (a) Show that $l = \{a + b\alpha + c\alpha^2 : a, b, c \in k\}$.
  (b) Find the minimal polynomial of each element in l over k.
  (c) Compute the group $Aut(l/k)$.

(5) Let k be a subfield of l and let $a \in l$ be transcendental over k.
  (a) Prove that $k(a)$ is isomorphic to $k(x)$ (the field of fractions of $k[x]$).
  (b) Prove that $k(a)$ is not finitely generated as a ring over k, that is to say there do not exist elements $b_1, \dots b_n \in k(a)$ such that $k(a) = k[b_1, \dots b_n]$. Hint: There are infinitely many irreducibles in $k[x]$.

(6) Use the Eisenstein criterion to prove that for any prime p, the polynomial $\frac{(x+1)^p - 1}{x}$ is irreducible.

(7) Let $p > 2$ be prime and let $\beta = \exp(2\pi i/p)$, so that β is an element of order p in the multiplicative group of the complex numbers. Find the minimal polynomial of β over $\mathbb{Q}$.