MATH STUDIES ALGEBRA SPRING 2018: HOMEWORK 4

JC

This homework is due by class time on Monday 19 February. It must be typeset (preferably in LaTeX) and submitted as a PDF file on the Canvas site, with a filename of the form

`andrewID_alg_homeworknumber.pdf`

For each minute that it is late, the grade will be reduced by 10 percent.

(1) Let A be an $n \times n$ matrix with integer entries. Let G be the subgroup of $\mathbb{Z}^n$ generated by the columns of A.
  (a) Prove that the rank (in the sense of $\mathbb{Z}$-modules) of G is equal to the rank (in the sense of matrices with real entries) of the matrix A.
  (b) Prove that $\mathbb{Z}^n/G$ is finite if and only if A is non-singular, and in this case $|\mathbb{Z}^n/G| = |\det(A)|$.

(2) Let R be a commutative ring with 1, let I be an ideal of R and let M be an R-module. Let IM be the subset of M consisting of elements of the form $\sum_{i=1}^n r_i m_i$ for $r_i \in I$, $m_i \in M$.
  (a) Prove that IM is a submodule of M.
  (b) Prove that if we attempt to define a scalar multiplication map from $R/I \times M/IM$ to M/IM by $(r+I)(m+IM) = rm + IM$, then we get a well-defined map which makes M/IM into an R/I-module.
  (c) Prove that if R is not the zero ring and R^m is isomorphic to R^n as an R-module for positive integers m and n, then $m = n$. Hint: R has at least one maximal ideal.

(3) Let R be a commutative ring with 1 and let M be an R-module such that $M \oplus N$ is free for some R-module N. Prove that if $\alpha : N_1 \to N_2$ is any surjective R-linear map, then $Hom(M, \alpha) : Hom(M, N_1) \to Hom(M, N_2)$ is also surjective. Hint: Start with the easier case where M itself is free.

(4) Fill in the steps in the following outline of a proof that if R is a PID, N is a free R-module (possibly of infinite rank) and $M \leq N$ then M is also free. You should add a proof for each claim (many of them will be short).
  (a) Claim 1: N has a basis, say $\{e_i : i \in I\}$ for some index set I.
  (b) Claim 2: The set I has a well-ordering $<_I$.
  (c) Claim 3: There is a linear map $p_i : N \to R$ such that $p_i(e_j) = 1$ for $i = j$ and 0 otherwise.
  (d) Claim 4: If $N_j = span\{e_i : i \leq_I j\}$, then $N_j \cap M \leq N$.
  (e) Claim 5: $p_j[N_j \cap M]$ is an ideal of R.
  (f) Claim 6: $p_j[N_j \cap M] = Ra_j$ for some $a_j \in R$.
  (g) Claim 7: There is $m_j \in N_j \cap M$ such that $p_j(m_j) = a_j$.
  (h) Claim 8: $\{m_j : a_j \neq 0\}$ is linearly independent. Hint: use the p_j's and the fact that any finite subset of I has a largest element.

(i) Claim 9: $\{m_j : a_j \neq 0\}$ spans M. Hint: Given nonzero $n \in N$, define $i(n)$ to be the largest $i \in I$ such that e_i appears with nonzero coefficient in the expansion of n. If the given set is not spanning, start by choosing $m \in M$ outside the span with $i(m)$ minimal.

(5) An $n \times n$ real matrix A is said to be *skew-symmetric* if $A + A^T = 0$, and *orthogonal* if $AA^T = I$.

(a) Let $A(t)$ be a differentiable matrix-valued function (taking values in the set of $n \times n$ real matrices) such that $A(t)$ is orthogonal for all t and $A(0) = I$. Prove that $A'(0)$ is skew-symmetric.

(b) Prove that if B is skew-symmetric and $A(t) = \exp(Bt)$, then $A(t)$ is orthogonal for all t and $A'(0) = B$.