MATH STUDIES ALGEBRA SPRING 2018: HOMEWORK 2

JC

This homework is due by class time on Monday 5 February. It must be typeset (preferably in LaTeX) and submitted as a PDF file on the Canvas site, with a filename of the form

`andrewID_alg_homeworknumber.pdf`

For each minute that it is late, the grade will be reduced by 10 percent.

(1) Let R be a ring with 1, and consider a sequence of R-modules $(M_i)_{i \in I}$ for some interval I in $\mathbb{Z}$, together with morphisms $\phi_i : M_i \to M_{i+1}$ defined when $i, i+1 \in I$. Such a sequence is said to be *exact at* i if $i-1, i, i+1 \in I$ and the image of ϕ_{i-1} is equal to the kernel of ϕ_i, and to be *exact* if it is exact at each relevant i.

Note: usually we describe exact sequences informally, by writing things like

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$

Also we typically don't name the arrows when it is clear what they must be. In particular the zero module 0 is both initial and terminal, so we just write $0 \to M$ or $N \to 0$.

Prove that:

(a) The sequence

$$0 \longrightarrow B \xrightarrow{\alpha} C$$

is exact iff α is injective.

(b) The sequence

$$A \xrightarrow{\alpha} B \longrightarrow 0$$

is exact iff α is surjective.

(c) The sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \longrightarrow 0$$

is exact iff α is an isomorphism.

(d) If the sequence

$$0 \longrightarrow A \xrightarrow{\alpha} B \xrightarrow{\beta} C \longrightarrow 0$$

is exact, then $C \simeq B/\alpha[A]$.

(2) Let H be a non-trivial torsion-free abelian group, and assume that H has the following property: for all nonzero elements $a, b \in H$ there exist nonzero

integers m, n such that $ma = nb$ (groups with this property are sometimes said to have "rank one").

For any nonzero $a \in H$ and any prime number p, define $n_p(a)$ as follows: $n_p(a)$ is the largest integer $k \geq 0$ such that there exists b with $p^k b = a$ (b will necessarily be unique as H is torsion-free), or $n_p(a) = \infty$ if such b exists for all k.

Let $a, b \in H$ be nonzero. Prove that $n_p(a) = \infty$ if and only if $n_p(b) = \infty$, and also that $\{p : n_p(a) \neq n_p(b)\}$ is finite.

Hint: facts about gcd's may be helpful.

(3) Recall that a subset $A \subseteq \mathbb{R}$ is *open* if for all $a \in A$ there is $\epsilon > 0$ such that $(a - \epsilon, a + \epsilon) \subseteq A$, and *closed* if its complement is open. It is a standard fact that a subset C of $\mathbb{R}$ is closed if and only if every convergent sequence (x_n) with $x_n \in C$ converges to a point of C. It is also standard that if B is a subset of $\mathbb{R}$ and C is the set of limits of convergent sequences of elements of B, then C is closed and is the least closed set containing B: in this case we write $C = \bar{B}$ and call C the *closure* of B.
  (a) Give an example of a non-closed subgroup of $(\mathbb{R}, +)$.
  (b) Prove that the closure of a subgroup of $(\mathbb{R}, +)$ is also a subgroup.
  (c) Prove that a non-trivial closed subgroup of $(\mathbb{R}, +)$ must either be an infinite cyclic group or be $\mathbb{R}$ itself.
  (d) (Challenging, not for credit) Describe with proof the closed subgroups of $(\mathbb{R}^2, +)$

(4) Let R be a commutative ring with 1, let M be an R-module, and recall from class the definition of the functor $Hom(M, -)$.

Give a proof or a counterexample for each of the following statements:
  (a) If $\alpha : N_1 \to N_2$ is injective, then $Hom(M, \alpha) : Hom(M, N_1) \to Hom(M, N_2)$ is injective.
  (b) If $\alpha : N_1 \to N_2$ is surjective, then $Hom(M, \alpha) : Hom(M, N_1) \to Hom(M, N_2)$ is surjective.
  (c) If $\alpha : N_1 \to N_2$ is an isomorphism, then $Hom(M, \alpha) : Hom(M, N_1) \to Hom(M, N_2)$ is an isomorphism.

(5) A function from $\mathbb{R}$ to the set of real $N \times N$ matrices is differentiable if each of the N^2 functions corresponding to the entries is differentiable, and the derivative is the matrix made up of the derivatives of the entries. Let A be a real $N \times N$ matrix. Prove that the matrix exponential function $\exp(At)$ is a differentiable function of t, and find its derivative.

Hint 1: since $AtAh = AhAt$ you can simplify the expression $\frac{\exp(At+Ah)-\exp(At)}{h}$ in a helpful way.

Hint 2: You may find it useful to bound the entries in $\sum_{i=2}^{\infty} \frac{(Ah)^i}{i!}$.