MATH STUDIES ALGEBRA SPRING 2018: HOMEWORK 1

JC

This homework is due by class time on Monday 29 January. It must be typeset (preferably in LaTeX) and submitted as a PDF file on the Canvas site, with a filename of the form

`andrewID_alg_homeworknumber.pdf`

For each minute that it is late, the grade will be reduced by 10 percent.

(1) Let G be a finite group, let k be a field and let R be the set of all functions from G to k. In a suggestive notation we will write a typical element $f \in R$ in the form $\sum_{i=1}^n a_i g_i$, where $g_1, \dots g_n$ enumerates G and $f(g_i) = a_i$.

We define operations $+$ and $\times$ on R as follows:

- $\sum_{i=1}^n a_i g_i + \sum_{i=1}^n b_i g_i = \sum_{i=1}^n c_i g_i$, where $c_i = a_i + b_i$.
- $\sum_{i=1}^n a_i g_i \times \sum_{j=1}^n b_j g_j = \sum_{k=1}^n c_k g_k$, where $c_k = \sum \{a_i b_j : g_i g_j = g_k\}$.

(a) Prove that R forms a ring with 1, given these definitions of $+$ and $\times$.
(b) Prove that $(R, \times)$ contains an isomorphic copy of G.
(c) Prove that R contains an isomorphic copy of k.
(d) Identify (with proof) the class of groups G for which R is commutative.

(2) Let k be a field, G a finite group and V a vector space over k (IE a k-module). Suppose that G acts on V in such a way that the map $v \mapsto g \cdot v$ is linear for each $g \in G$. Let R be the ring constructed from k and G in the preceding question.

(a) Prove that if we define $(\sum_{i=1}^n a_i g_i) v = \sum_{i=1}^n a_i (g_i \cdot v)$, then V becomes a left R-module.
(b) Prove that all left R-modules arise in this way.

(3) Consider the ring R of $N \times N$ real matrices. We will generalise some familiar ideas from calculus to this setting:

- If $(M_n)_{nin\mathbb{N}}$ is a sequence of elements of R, then (M_n) is *convergent* if and only if for each i and j the sequence formed by the (i,j)-entries in M_n is convergent. In this case the *limit* is the matrix M whose (i,j)-entry is the limit of the sequence formed by the (i,j)-entries in M_n.
- Similarly, an infinite sum $\sum_{n=0}^\infty M_n$ of elements of R is convergent iff the sequence of partial sums $\sum_{n=0}^j M_n$ is convergent, and in this case $\sum_{n=0}^\infty M_n$ is defined as the limit of the sequence of partial sums.

(a) Prove that for any matrix $A \in R$, the power series $\sum_{n=0}^\infty \frac{A^n}{n!}$ converges. Hint: Let K be the maximum of the absolute values of the entries in A. Find an upper bound for the absolute values of the entries in A^n in terms of K, N and n. Now use the Comparison Test.
(b) Compute the sum of the series $\sum_{n=0}^\infty \frac{A^n}{n!}$ when $N = 2$ and $A = \begin{pmatrix} \lambda & 0 \\ 1 & \lambda \end{pmatrix}$

(c) (Challenging, not for credit) We define $\exp(A) = \sum_{n=0}^{\infty} \frac{A^n}{n!}$. Prove that if $AB = BA$ then $\exp(A + B) = \exp(A)\exp(B)$.

(4) Review the following basic definitions from category theory: *category*, *object*, *morphism*, *initial object*, *terminal object*.

(5) Let $\mathcal{C}$ be a category, let I be some (potentially infinite) set I and let $(a_i)_{i \in I}$ be an I-indexed family of objects of $\mathcal{C}$. A *cone* over this family consists of an object b and an I-indexed family of morphisms $(f_i : b \to a_i)_{i \in I}$ (I usually imagine (a_i) as a horizontal line of objects and b sitting above them all with arrows going down into each a_i)

We will make a new category $\mathcal{D}$ as follows: The objects of $\mathcal{D}$ are cones over (a_i). If B is a cone consisting of an object b and morphisms $(f_i : b \to a_i)_{i \in I}$, and C consists of an object c and morphisms $(g_i : c \to a_i)_{i \in I}$, then a morphism from B to C in $\mathcal{D}$ is a morphism h from b to c in $\mathcal{C}$ such that $f_i = g_i \circ h$.

(a) Verify that $\mathcal{D}$ is a category.

(b) A *product* for the sequence $(a_i)_{i \in I}$ is a terminal object in the category $\mathcal{D}$. Prove that if $\mathcal{C}$ is the category of left R-modules over a ring R then every sequence has a product.

(6) Let G be an abelian group. A *torsion element* of G is an element $g \in G$ such that $ng = 0$ for some integer $n > 0$, and G is *torsion-free* if 0 is the only torsion element.

(a) Let G be any abelian group. Prove that the set H of torsion elements forms a subgroup of G, and that the quotient G/H is torsion-free.

(b) Given an example of an infinite abelian group in which every element is a torsion element.

(c) Let G be a torsion-free abelian group. Prove that there is a vector space V over the field $\mathbb{Q}$ such that G is a subgroup of $(V, +)$ and every element of V has the form $\frac{1}{n}g$ for some integer $n > 0$ and $g \in G$ (n and g will not be unique here).