MATH STUDIES HOMEWORK 6

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Submit the LaTeX file containing your solutions to the email address algebrahw@legba.math.cmu.edu by 1159 pm on Friday 8 April.

(1) Let V be a separable complex Hilbert space. Show that if $T \in B(V)$ and V has an ON Schauder basis consisting of eigenvectors of T, then $TT^\dagger = T^\dagger T$.

(2) Let V be a separable complex Hilbert space and let $A \in B(V)$ be self-adjoint. Show that $U = (A - iI)(A + iI)^{-1}$ is UNITARY.

(3) Let V be a separable complex Hilbert space and let M and N be vector subspaces such that $M + N = V$ and $M \perp N$. Show that M and N are closed.

(4) Let V be a separable complex Hilbert space and let $T \in B(V)$. Prove that if $T^\dagger T$ is compact then T is compact.

(5) Let (A_n) be an infinite sequence of $m \times m$ complex matrices. We will say that the sequence converges to a matrix B if and only if the sequence (a^n_{ij}) converges to b_{ij} for all i and j with $1 \leq i, j \leq m$, where $a^n_{i,j}$ is the (i, j) entry of A_n.

Similarly an infinite sum of $m \times m$ matrices $A_0 + A_1 + A_2 \ldots$ converges iff the sequence of partial sums $A_0, A_0 + A_1, A_0 + A_1 + A_2, \ldots$ converges.

(a) Show that for any complex $m \times m$ matrix A the "exponential series"

$$exp(A) = I + A + A^2/2! + A^3/3! + \ldots$$

converges. Hint: let $\|A\|$ denote the max of the absolute values of the entries of A, and estimate $\|A^i\|$.

(b) Derivatives of matrix valued functions are defined in the obvious ("entry by entry") way. Show that the derivative of the matrix function $exp(tA)$ with respect to t is $Aexp(tA)$.

(c) Show that if $AB = BA$ then $exp(A + B) = exp(A)exp(B)$. Is this true in general?

(d) Show that if the trace (sum of the diagonal entries) of A is zero, then $exp(A)$ has determinant one.

(e) Show that if A is Hermitian then $exp(iA)$ is unitary.