MATH STUDIES HOMEWORK 5

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Submit the LaTeX file containing your solutions to the email address algebrahw@legba.math.cmu.edu by 1159 pm on Friday 25 March.

A point of clarification: when we say that an operator K in $B(V)$ is invertible we mean that there is L in $B(V)$ such that $KL = LK = I$ where as usual I is the identity.

As we see in class soon, if a map in $B(V)$ is bijective then its inverse is bounded. So a non-invertible K in $B(V)$ fails to be bijective, hence either it's not into or not onto. If V is FD then not-into equals not-onto but they are distinct in l_2; consider the maps taking $(x_0, x_1, x_2 \ldots)$ to $(0, x_0, x_1, \ldots)$ and $(x_1, x_2, ldots)$.

(1) Let H be the subset of $l_2(\mathbb{C})$ consisting of those sequences with $|x_n| \leq 1/(n+1)$. Show that H is compact. Hint: sequential compactness.

(2) Let K be the bounded operator on $l_2(\mathbb{C})$ which maps (x_n) to $(x_n/(n+1))$. Show that K is compact. (Trickier) For which sequences of complex numbers (α_n) is the map taking (x_n) to $(\alpha_n x_n)$ compact?

(3) Let K be as in the previous question. Find the eigenvalues of K and show that 0 is in the spectrum but not an eigenvalue. Is 0 the only point in the spectrum which is not an eigenvalue?

(4) Let $V = C^n$ and let T be a linear operator on V. Show that $TT^\dagger$ is self-adjoint. Are all self-adjoint operators of this form?

(5) Let V and W be complex Hilbert spaces. Show how to give $V \oplus W$ a Hilbert space structure so that $V \oplus \{0\}$ is a closed subspace IMic to V and $\{0\} \oplus W$ is a closed subspace IMIc to W. Can you find a reasonable Hilbert space structure for $V \otimes W$?