MATH STUDIES HOMEWORK 3

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Submit the LaTeX file containing your solutions to the email address algebrahw@legba.math.cmu.edu by 1159 pm on Fri 18 Feb.

(1) Halmos page 123 question 4 (don't miss out part e!)

(2) Recall that $l_2(\mathbb{R})$ is the space of all infinite sequences (x_n) such that $x_n \in \mathbb{R}$ and $\sum_n x_n^2 < \infty$, equipped with an inner product by writing $\langle x, y \rangle = \sum_n x_n y_n$. We defined the length or norm of x to be $\|x\| = \langle x, x \rangle^{1/2}$ and saw that if we set $d(x, y) = \|x - y\|$ then we get a metric.

Show that the "unit ball" $\{x : \|x\| \leq 1\}$ is not compact in this metric.

(3) Show that the kernel of a continuous linear map from $l_2(\mathbb{R})$ to $l_2(\mathbb{R})$ is closed.

(4) Find a linear map from $l_2(\mathbb{R})$ to $l_2(\mathbb{R})$ which is not continuous.

(5) Show that by contrast if we give $\mathbb{R}^n$ the usual Euclidean metric then all linear maps from $\mathbb{R}^n$ to $\mathbb{R}^n$ are continuous.

(6) Let $0 < q < p < r$, and let a_i and μ_i be positive reals for $1 \leq i \leq n$. Show that

$$\sum_{i-1}^{n} \mu_i a_i^p \leq \left(\sum_{i=1}^{n} \mu_i a_i^q \right)^{(r-p)/(r-q)} \left(\sum_{i=1}^{n} \mu_i a_i^r \right)^{(p-q)/(r-q)}$$