MATH STUDIES HOMEWORK 2

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Submit the LaTeX file containing your solutions to the email address algebrahw@legba.math.cmu.edu by 1159 pm on Fri 4 Feb.

(1) Let U, V and W be VS's (not necessarily finite dimensional) over a field F. Show that the spaces $Hom(U, Hom(V, W))$ and $Hom(U \otimes V, W)$ are isomorphic *without* choosing bases.

(2) Let V be a VS over F of dimension m. Show that if $v_1, \ldots v_m \in V$ then the v_i are independent iff $v_1 \wedge \ldots v_m \neq 0$ in the exterior power $\bigwedge^m V$. Hint: you may wish to choose a basis $w_1, \ldots w_m$ and consider the unique linear map T taking w_i to v_i.

(3) Let E and F be fields with E a subfield of F, and let V be a VS over E.

  (a) Show that F is a VS over E if we define $+$ on F to be the field addition in F, and define scalar multiplication by the restriction of field multiplication in F to $E \times F$.

  (b) Let $W = F \otimes_E V$, that is the E-VS obtained by forming the tensor product of the E-VS's F and V.

    (i) Show that if $\lambda \in F$ then the map from $F \times V$ to W given by $(f, v) \mapsto (\lambda f) \otimes v$ is a bilinear map of E-VS's. Deduce that there is a unique E-linear map ψ_λ from W to W such that $\psi_\lambda(f \otimes v) = (\lambda f) \otimes v$ for all f and v.

    (ii) Show that if we define a scalar multiplication map from $F \times W$ to W by $\lambda w = \psi_\lambda(w)$ then W becomes an F-VS.

  (c) Show that there is an F-subspace X of W which (when we regard it as a E-VS by restricting the scalar multiplication map to $E \times X$) is isomorphic to V.

(4) Let $\mathcal{C}$ be a category and let a and b be objects of $\mathcal{C}$. Say that a *diagram* over $\{a, b\}$ is a pair D of morphisms (f, g) where $f : c \to a$ and $g : c \to b$ for some object c of $\mathcal{C}$.

  (a) Show that we can construct a category $\mathcal{D}$ such that the objects of $\mathcal{D}$ are the diagrams over $\{a, b\}$, and the morphisms in $\mathcal{D}$ from the diagram (f, g) to the diagram (f', g') are the morphisms h of $\mathcal{C}$ such that $f' \circ h = f$ and $g' \circ h = g$.

(b) A *product* of a and b is a diagram (f, g) over $\{a, b\}$ such that for any diagram (f', g') over $\{a, b\}$ there is a unique morphism h of $\mathcal{C}$ such that $f' = f \circ h$ and $g' = g \circ h$. Show that if (f_1, g_1) and (f_2, g_2) are both products of $\{a, b\}$ there is a unique isomorphism h such that $f_2 \circ h = f_1$ and $g_2 \circ h = g_1$.

(c) Show that products always exist in the category of groups and group HMs, and the category of sets and functions. Find a category in which not all products exist.

(5) Let V be a space of dimension m and let $n \leq m$. Show that the exterior power $\bigwedge^n V$ has dimension $\binom{m}{n}$.