MATH STUDIES HOMEWORK 2

JAMES CUMMINGS

Submit the \LaTeX file containing your solutions to the email address algebrahw@legba.math.cmu.edu by 1159 pm on Fri 4 Feb.

(1) Let $U, V$ and $W$ be VS’s (not necessarily finite dimensional) over a field $F$. Show that the spaces $\text{Hom}(U, \text{Hom}(V,W))$ and $\text{Hom}(U \otimes V, W)$ are isomorphic without choosing bases.

(2) Let $V$ be a VS over $F$ of dimension $m$. Show that if $v_1, \ldots, v_m \in V$ then the $v_i$ are independent iff $v_1 \wedge \ldots v_m \neq 0$ in the exterior power $\bigwedge^m V$. Hint: you may wish to choose a basis $w_1, \ldots w_m$ and consider the unique linear map $T$ taking $w_i$ to $v_i$.

(3) Let $E$ and $F$ be fields with $E$ a subfield of $F$, and let $V$ be a VS over $E$.
   (a) Show that $F$ is a VS over $E$ if we define $+$ on $F$ to be the field addition in $F$, and define scalar multiplication by the restriction of field multiplication in $F$ to $E \times F$.
   (b) Let $W = F \otimes_E V$, that is the $E$-VS obtained by forming the tensor product of the $E$-VS’s $F$ and $V$.
      (i) Show that if $\lambda \in F$ then the map from $F \times V$ to $W$ given by $(f, v) \mapsto (\lambda f) \otimes v$ is a bilinear map of $E$-VS’s. Deduce that there is a unique $E$-linear map $\psi_\lambda$ from $W$ to $W$ such that $\psi_\lambda(f \otimes v) = (\lambda f) \otimes v$ for all $f$ and $v$.
      (ii) Show that if we define a scalar multiplication map from $F \times W$ to $W$ by $\lambda w = \psi_\lambda(w)$ then $W$ becomes an $F$-VS.
   (c) Show that there is an $F$-subspace $X$ of $W$ which (when we regard it as a $E$-VS by restricting the scalar multiplication map to $E \times X$) is isomorphic to $V$.

(4) Let $\mathcal{C}$ be a category and let $a$ and $b$ be objects of $\mathcal{C}$. Say that a diagram over $\{a, b\}$ is a pair $D$ of morphisms $(f, g)$ where $f : c \rightarrow a$ and $g : c \rightarrow b$ for some object $c$ of $\mathcal{C}$.
   (a) Show that we can construct a category $\mathcal{D}$ such that the objects of $\mathcal{D}$ are the diagrams over $\{a, b\}$, and the morphisms in $\mathcal{D}$ from the diagram $(f, g)$ to the diagram $(f', g')$ are the morphisms $h$ of $\mathcal{C}$ such that $f' \circ h = f$ and $g' \circ h = g$. 

1
(b) A product of $a$ and $b$ is a diagram $(f, g)$ over $\{a, b\}$ such that for any diagram $(f', g')$ over $\{a, b\}$ there is a unique morphism $h$ of $C$ such that $f' = f \circ h$ and $g' = g \circ h$. Show that if $(f_1, g_1)$ and $(f_2, g_2)$ are both products of $\{a, b\}$ there is a unique isomorphism $h$ such that $f_2 \circ h = f_1$ and $g_2 \circ h = g_1$.

(c) Show that products always exist in the category of groups and group HMs, and the category of sets and functions. Find a category in which not all products exist.

(5) Let $V$ be a space of dimension $m$ and let $n \leq m$. Show that the exterior power $\wedge^n V$ has dimension $\binom{m}{n}$.