LINEAR ALGEBRA HOMEWORK 1 SOLUTIONS

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Total points: A1=10, A2 = 10, A3 = 10, B1 = 5, B2 = 10, c1 = 5, c2 = 5, C3 =20 for a total of 75

Common problems flagged by the grader:

(1) Using wrong formula for matrix multiplication and stating that AB and BA. have exactly the same diagonal elements.
(2) Stating in A3 that the determinant of $A + \lambda B$ would be a polynomial of degree n without using the fact that B doesn't have any zeroes on its diagonal.
(3) Showing a case with $< n$ solutions as counter-example in A3. The question asked to prove that $\leq n$ solutions were possible, so a counterexample should show $> n$ solutions.
(4) Not stating that continuous functions on $\mathbb{R}$ are closed under linear combinations. They have stated that two real numbers when added yield a real number, which is not what closure means for problem C1.
(5) Many have included the diagonal elementary matrices in the basis for skew-symmetric matrices, which is wrong because such matrices are zero down the diagonal.
(6) Not arguing that the given subspaces are nonempty in C3.

A: Review of real vectors and matrices

A1. (10 points total) Recall that if x and y are vectors in $\mathbb{R}^n$, then the *inner product (dot product)* of x and y is $x \cdot y = \sum_{i=1}^n x_i y_i$.

Prove that if $v_1, \ldots v_{n+1} \in \mathbb{R}^n$ are vectors such that $v_i \cdot v_j = 0$ for $i \neq j$, then $v_i = 0$ for at least one value of i.

Suppose for a contradiction that they are all nonzero. We claim that they are linearly independent: for if $\sum_{i=1}^{n+1} \lambda_i v_i = 0$, then for each j we have

$$0 = v_j \cdot (\sum_{i=1}^{n+1} \lambda_i v_i) = \sum_i \lambda_i (v_j . v_i) = \lambda_j (v_j \cdot v_j).$$

and also $v_j \cdot v_j \neq 0$ so that $\lambda_j = 0$.

This is an immediate contradiction since $\mathbb{R}^n$ has dimension n.

There are several alternative ways to present the same idea, for example you can say that some v_i depends on the others and then dot the equation expressing this fact with v_i.

A2. (10 points total) Let $A = (a_{ij})$ be an $n \times n$ real matrix. The *trace of A* is defined by $tr(A) = \sum_{i=1}^n a_{ii}$. Prove that if A and B are $n \times n$ matrices and $C = AB - BA$ then $tr(C) = 0$. Can every matrix with trace zero be written in this form?

Let $A = (a_{ij})$ and $B = (b_{ij})$. Then $C = (c_{ik})$ where $c_{ik} = \sum_j (a_{ij} b_{j_k} - b_{ij} a_{jk})$, in particular $c_{ii} = \sum_j (a_{ij} b_{j_i} - b_{ij} a_{ji})$. So $tr(C) = \sum_{i,j} a_{ij} b_{ji} - \sum_{i,j} b_{ij} a_{ji} = 0$.

The last part was too hard, I should have asked for the easier fact that any traceless matrix is a sum of matrices of the form $AB - BA$. It its actually true that any traceless matrix is of form $AB - BA$ but the proof is very tricky, see

`http://projecteuclid.org/download/pdf_1/euclid.mmj/1028990168`

A3. (10 points total) Let $n > 0$ and let A and B be real $n \times n$ matrices with B diagonal (that is to say that all off-diagonal entries are zero). Prove that if B has all diagonal entries non-zero, then there are at most n real numbers λ such that $A + \lambda B$ is not invertible. Show by means of an example that in general the condition "B has at least one non-zero diagonal entry" is not strong enough to prove this conclusion.

Hint: What do you know about the number of roots of a polynomial in relation to its degree?

$A + \lambda B$ is not invertible ("singular") if and only if its determinant is zero. It follows from the determinant formula that $det(A + \lambda B)$ is a polynomial of degree n in λ, and such a polynomial has at most n zeroes.

For the last part, just observe that if $A = B = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ then $det(A + \lambda B) = 0$ for all λ.

B: Fields

B1. (5 points total) Define operations $+$ and $\times$ on $\mathbb{R}^2$ coordinatewise, that is $(a_1, a_2) + (b_1, b_2) = (a_1 + b_1, a_2 + b_2)$ and $(a_1, a_2) \times (b_1, b_2) = (a_1 \times b_1, a_2 \times b_2)$ Prove that this structure is not a field, and state which axioms fail.

The only axiom which fails is the one which states that every nonzero element has a multiplicative inverse. To see this observe that the "0" (identity for $+$) is $(0, 0)$ and the "1" (identity for $\times$) is $(1, 1)$. The element $(1, 0)$ is not $(0, 0)$ yet fails to have an inverse.

B2. (10 points total) Prove that if F is a field and $a \in F$ with $a \neq 0$, then $a^n \neq 0$ for all $n > 0$. Prove further that if F is finite then there exists $n > 0$ such that $a^n = 1$. Hint for the second part: consider the sequence $a, a^2, a^3, \ldots$

Since $a \neq 0$ it has an inverse a^{-1}, and clearly $a^{n+1} = 0 \implies a^{-1} a^{n+1} = a^n = 0$ or equivalently $a^n \neq 0 \implies a^{n+1} \neq 0$. Then an easy induction shows $a^n \neq 0$ for all n.

For the second part: since F is finite there must be a repetition say $a^m = a^n$ for $m < n$. Then $a^m(a^{n-m} - 1) = 0$, and since $a^m \neq 0$ we must have $a^{n-m} = 1$.

C: Vector spaces

C1. (5 points total) Let V be the set of all continous functions from $\mathbb{R}$ to $\mathbb{R}$. Define an operation $+$ on V by "pointwise addition", that is $f + g$ is the function such that $(f + g)(x) = f(x) + g(x)$ for all real x. Define a scalar multiplication by elements of $\mathbb{R}$ in the natural way, that is rf is the function such that $(rf)(x) = rf(x)$ for all x. Show that this makes V into a vector space over $\mathbb{R}$.

Routine: Just use the facts from elementary calculus that the sum or product of continuous functions is continuous.

C2. (5 points total) Let V be the set of all infinite sequences $(r_0, r_1, r_2, \ldots)$ such that

(a) $r_i \in \mathbb{R}$ for all $i \in \mathbb{N}$.

(b) There is $m \in \mathbb{N}$ such that $r_n = 0$ for all $n \geq m$.

Define

$$(r_0, r_1, r_2, \ldots) + (s_0, s_1, s_2, \ldots) = (r_0 + s_0, r_1 + s_1, \ldots),$$

and

$$\lambda(r_0, r_1, r_2, \ldots) = (\lambda r_0, \lambda r_1, \lambda r_2, \ldots).$$

Do these definitions make V into a vector space over $\mathbb{R}$?

Routinely the answer is YES. The main point is that V is closed under the addition and scalar multiplication operations.

C3. (20 points total) Let $V = M_{n\times n}(\mathbb{R})$, that is the set of $n \times n$ real matrices. It is easy to see that V is a real vector space with the usual notions of adding two matrices and multiplying a matrix by a real number (you need not prove this).

(a) (6 points for this part) Prove that each of the following is a subspace of V: the set of symmetric matrices $\{A : A = A^T\}$, the set of skew-symmetric matrices $\{A : A = -A^T\}$ and the set of diagonal matrices. Each of these sets is nonempty and closed under linear combinations.

(b) Give a basis for V and for each of the subspaces of V listed in the previous part. (8 points for this part)

Let $e_{i,j}$ be the matrix with a 1 at the i row and j column and 0's elsewhere.

Then:

(i) The set of all $e_{i,j}$ is a basis for V.

(ii) The set of matrices which are of form $e_{i,i}$ or $e_{i,j} + e_{j,i}$ $(i < j)$ form a basis for the space of symmetric matrices.

(iii) The set of matrices which are of form $e_{i,j} - e_{j,i}$ $(i < j)$ form a basis for the space of skew-symmetric matrices.

(iv) The set of matrices which are of form $e_{i,i}$ form a basis for the space of diagonal matrices.

(c) (6 points for this part) Let B be an $n \times n$ real matrix and let $W_B = \{A : AB = 0\}$. Prove that W_B is a subspace of V.

This bit is routine: the zero matrix is in W_B and it is closed under linear combinations.

Prove or disprove: "for every integer d with $0 \leq d \leq n^2$ there is a matrix B such that W_B has dimension d". Hint: $A \in W_B$ if and only if every row of A is in the space of row vectors v such that $vB = 0$.

Let V_0 be the space of row vectors v with $vB = 0$, it has some dimension m where $0 \leq m \leq n$ (actually $vB = 0 \iff B^T v^T = 0$, so m is the nullity of B^T which of course is the nullity of B).

Let us fix a basis $v_1, \ldots v_m$ for V_0. Now we use this to cook up a basis for W_B consisting of matrices $C_{i,j}$ for $1 \leq i \leq m$ and $1 \leq j \leq n$: $C_{i,j}$ is the matrix where row j is v_i and all other entries are zero.
So the dimension of W_B is mn, and for $n > 1$ this means that not all values d can be attained.