CA LECTURE 29

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Let C be some class of R-modules (typically it will be the class of fg R-modules or maybe N'ian R-modules, to be on the safe side let us demand that C is closed under taking submodules and quotients). An *additive function on* C is a map $\lambda : C \to \mathbb{Z}$ such that for any short exact $0 \to L \to M \to N \to 0$ where the modules are in C, $\lambda(L) - \lambda(M) + \lambda(N) = 0$.

Example: C is FDVS's over a field F and λ is the dimension function.

Remark: Easy to see that $\lambda(0) = 0$, and $\lambda(M)$ depends only on the isomorphism class of M.

Lemma: If $0 \to M_1 \to M_2 \ldots \to M_n \to 0$ is exact with the $M_i \in C$ then $\sum_{i=1}^{n}(-1)^i \lambda(M_i) = 0$.

Proof: Let $\alpha_i : M_i \to M_{i-1}$ Apply the additivity of λ to each of the associated short exact $0 \to ker(\alpha_i) \to M_i \to im(\alpha_i) = ker(\alpha_{i+1}) \to 0$, and combine so that the terms from the kernels cancel out.

Def: A module M is simple iff $M \neq 0$, and the only submodules are 0 and M.

Easy to see: if $M \leq M'$ then M'/M is simple iff for all L with $M \leq L \leq M'$ either $L = M$ or $L = M'$.

Def: A *(strict) chain of length* k of submodules of a module M is a sequence $N_0 \supsetneq N_1 \supsetneq N_2 \ldots N_k$ of submodules of M. Such a chain is a *composition series* for M iff $N_0 = M$, $N_k = 0$ and N_i/N_{i+1} is simple.

A little thought shows that a composition series is just a maximal chain.

Def: The *length* $l(M)$ of M is the least length of a composition series if one exists, or ∞ otherwise.

Note that $l(M) < \infty$ iff M has a compsoition series.

Theorem 1. *Let M be a module with $l(M) = k < \infty$. Then*

(1) *For every $N < M$, $l(N) < k$.*
(2) *Every strict chain in M has length at most k.*
(3) *The composition series are precisely the strict chains of length k.*
(4) *Every strict chain can be extended to a composition series.*

Proof. Fix a composition series of length k, $M_0 \supsetneq \ldots \supsetneq M_k$.

Let $N \leq M$ and define $N_i = M_i \cap N$. We may regard N_i/N_{i+1} as a submodule of M_i/M_{i+1} so by simpolicity either $N_i = N_{i+1}$ or N_i/N_{i+1} is simple. By deleting repetitions we get a composition series of length at most k for N. If the series has length k then $N_i \neq N_{i+1}$ for all i and an easy induction (backwards) shows that $N_i = M_i$ for all i, in particular $N = N_0 = M_0 = M$.

If $X_0 \supsetneq \ldots \supsetneq X_j$ is a strict chain then applying the last claim with $M = X_i$ and $N = X_{i+1}$ shows that $l(X_{i+1}) < l(X_i) \leq l(M) = k$. So easily $j \leq k$.

Any strict chain of length k can not be extended, so is a composition series. Conversely any composition series has length at least k by choice of k, hence length exactly k. Starting with any strict chain we may just add in modules till we get stuck, and obtain a composiiton series.

$\square$

Corollary: If M has a composition series it is both Noetherian and Artinin.

Fact: The converse is also true,

Proof: Start with $M_0 = M$ and use the N'ian property to choose M_{i+1} maximal under inclusion in $\{N : N < M_i\}$. This terminates by the A'ian property.

Fact: If R is A'ian and M is fg then $l(M) < \infty$.

Proof: R is both A'ian and N'ian, so M ditto.

Fact: The map $M \mapsto l(M)$ is additive on $\{M : l(M) < \infty\}$.

Proof: ETS it for $M \leq N$ and the sequence $0 \leq M \leq N \leq N/M \leq 0$. Given a composition series for M and one for N/M, lift the series for N/M to a series of submodules running from N down to M and then glue it to the composition series for M. Check it works.

Now we turn to the material of Chapter 11.

In a graded ring R with components R_n we say $r \in R_n$ is *homogeneous* and write $deg(r) = n$. A similar convention will apply to graded R-modules.

Let R be a graded N'ian ring. As we saw R_0 is N'ian subring of R and R is ring-finite. Choose homogeneous generators $h_1, \ldots h_k$ for R. Let M be a fg graded R-module and choose homogeneous generators $m_1, \ldots m_l$.

Since $R_0 M_n \subseteq M_n$ we see that M_n is an R_0-module. Considerations of grading imply that every element of M_n is an R_0-linear combination of elements of the form $h_1^{x_1} \ldots h_k^{x_k} m_j$ where $\sum_{i=1}^k x_i deg(h_i) + deg(m_j) = n$. In particular each M_n is fg as an R_0-module.

Let λ be an additive function on the class of fg R_0-modules. We define $P(M,t) = \sum_n \lambda(M_n) t^n$.

Next time we prove some remarkable facts:

Firstly $P(M,t)$ is a rational function of t, in fact one of the special form

$$\frac{f}{\prod_{i=1}^k (1 - t^{deg(h_i)})}$$

for some $f \in \mathbb{Z}[t]$.

Secondly if $deg(h_i) = 1$ for all i then there is polynomial $g \in \mathbb{Q}[x]$ of degree less than k such that $\lambda(M_n) = g(n)$ for all large n.