CA LECTURE 25 (SPECIAL HALLOWEEN EDITION)

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Once again all groups are abelian.

G is a (abelian) group. We fix a decreasing sequence of subgroups $G_0 \geq G_1 \geq \dots$ and generate a topology as follows: $O \subseteq G$ is open iff for every $a \in O$ there is n such that $a + G_n \subseteq O$. To put it another way the translates $a + G_n$ form a basis.

Example: if $G = (\mathbb{Z}, +)$ and $G_n = p^n\mathbb{Z}$ this is the topology from the p-adic metric discussed in the HW.

0.1. **Completion.** Definition: a sequence $(a_n)_{n\in\mathbb{N}}$ is *Cauchy* iff for all j there is m such that $a_n - a_m \in G_j$ for all $n \geq m$ (more transparently we could say that the coset $a_n + G_j$ is constant for large n). Two such sequences $(a_n), (b_n)$ are *equivalent* iff for all j there is m such that $a_n - b_n \in G_j$ for $n \geq m$.

Easy to see that the set of classes form a group under pointwise addition. The resulting group $\hat{G}$ is the *completion* of G.

Jeremy Bradford asked the excellent question "Does this not depend on the G_n?" Actually it only depends on the topology they induce, as we see soon on the HW.

We have a natural surjective inverse system of groups

$$G/G_0 \leftarrow G/G_1 \leftarrow G/G_2 \leftarrow \dots$$

where the map from G/G_n to G/G_m for $n \geq m$ is $g + G_n \mapsto g + G_m$.

It is easy (just like a recent HW) to see that the completion $\hat{G}$ is isomorphic to the inverse limit of this system.

Sketch of the proof: if (a_n) is Cauchy then by definition we may find (b_j) such that $b_j \in G/G_j$ and $a_n + G_j = b_j$ for all large n. Routine to check that $(b_j) \in \varprojlim G/G_j$ and that this map sets up an IM.

Now we fix G and the G_n and consdier an arbitrary subgroup $H \leq G$ and the associated quotient group G/H, as usual π_H is the projection map.

We may induce decreasing chains of subgroups $G_n \cap H$ and $\pi_H[G_n] = (G_n + H)/H$ in H and G/H.

A little bit of thoughts shows that the exact sequence $0 \to H \to G \to G/H \to 0$ induces an exact sequence $0 \to G_n \cap H \to G_n \to (G_n + H)/H \to 0$, which in turn gives us an exact sequence $0 \to H/(G_n \cap H) \to G/G_n \to (G/H)/((G_n + H)/H) \to 0$.

Think of this as row n in a big commutative diagram with the inverse systems defining the completions of H, G and G/H as the non-trivial columns. By a result from last time we get an exact sequence

$$0 \to \varprojlim H/(G_n \cap H) \to \varprojlim G/G_n \to \varprojlim (G/H)/((G_n + H)/H) \to 0.$$

Slogan: IF YOU CHOOSE THE RIGHT TOPOLOGIES, then completion preserves exact sequences.

Caution: In the intended application each module M has a natural topology (the I-topology) but it is false in general that when $M \leq M'$ the I-topology on M'

will induce as above the I-topology on M or on M'/M. This is the subject of the Artin-Rees lemma which we prove next time.

Important special case: let $H = G_m$. Then (as you can check!) we induce the subspace topology on G_m and the discrete topology on G/G_m. Completing wrt the discrete topology nothing happens so we get an exact sequence

$$0 \to \hat{G}_m \to \hat{G} \to \widehat{G/G_m} \simeq G/G_m \to 0,$$

and so $G/G_m \simeq \hat{G}/\hat{G}_m$.

In particular if we topologise $\hat{G}$ using the subgroups $\hat{G}_m$ and then complete, we get the inverse limit of $\hat{G}/\hat{G}_m$ which is isomorphic to $\hat{G}$.

Main example: I is an ideal of R. We define the *I-topology* or *I-adic topology* on R using the subgroups of $(R, +)$

$$I \supseteq I^2 \supseteq \dots$$

More generally given an R-module M we define the I-topology using

$$IM \supseteq I^2M \supseteq \dots$$

0.2. **Graded rings and modules.** To analyse the completion wrt the I-topology introduce graded rings.

Recall that a module M is the internal direct sum of some modules $M_i \leq M$ iff every element of M is uniquely $\sum_i m_i$ where $m_i \in M_i$ and almost all (that is all but finitely many) of the m_i are zero. Equivalently the map $(m_i) \mapsto \sum_i m_i$ is an IM from $\oplus_i M_i$ to M.

A *graded ring* is a ring R together with subgroups R_n of $(R, +)$ for $n \in \mathbb{N}$ such that R is the internal direct sum of the R_n, and $R_m R_n \subseteq R_{m+n}$ for all m and n. If R s such a ring then a *graded R-module* is an R-module M together with subgroups M_n of $(M, +)$ for $n \in \mathbb{N}$ such that M is the internal direct sum of the M_n, and $R_m M_n \subseteq M_{m+n}$ for all m and n.

Example: $R[x]$ is graded by the subgroups Rx^n.

Remark: if R is graded then in general R_n is just a subgroup of $(R, +)$. But R_0 is a subring.

Lemma: Let R be a graded ring. Then TFAE

(1) R is a N'ian ring.
(2) R_0 is a N'ian ring and R is ring-finite over R_0.

Proof: 2 implies 1 by the Basissatz.

For 1 implies 2 let R_+ be the subgroup generated by the R_n for $n > 0$. Easily it's an ideal and $R_0 \simeq R/R_+$.

R_+ is fg as an ideal of R, let $s_1, \dots s_m$ be generators. Then we claim that $R = R_0[s_1, \dots s_m]$.