CA LECTURE 23

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Limits ctd: we have a diagram (in the very general sense), that is a (covariant) functor F from $\mathbb{I}$ to $\mathbb{C}$ and have the notion of a *limit* for the diagram, that is a final object in the category of cones. The limit may not exist but as usual if it exists it is unique up to a unique IM.

Suppose that we also have a (covariant) functor $T : \mathbb{C} \to \mathbb{D}$. Then $T \circ F$ is a functor from $\mathbb{I}$ to $\mathbb{D}$ so we may think of it as a diagram in $\mathbb{D}$. What is more if we have a cone $c, \{f_a\}$ over F then it is routine to check that $Tc, \{Tf_a\}$ is a cone over $T \circ F$. Now the question arises: does the functor T preserve limits?

We will prove a basic result: if T has a left adjoint then T preserves limits. Recall that a left adjoint is a functor $S : \mathbb{D} \to \mathbb{C}$ such that we can set up a natural bijection between $Hom(d, Tc)$ and $Hom(Sd, c)$ for all objects c of $\mathbb{C}$ and d of $\mathbb{D}$. If the naturality conditions looked mysterious to you then the following proof should provide motivation for them.

In the handout where these ideas were introduced we presented the naturality condition as one equation. Here it is more perspicuous to present the condition in an equivalent form as two equations. To lighten the notation if $f : d \to Tc$ in $\mathbb{D}$ we write $f^* : Sd \to c$ for the corresponding morphism in $\mathbb{C}$ (of course the map $f \mapsto f^*$ depends on c and d)

Naturality condition N1: For all c' and all $g : c \to c'$, $(Tg \circ f)^* = g \circ f^*$.

Naturality condition N2: If $h : d' \to d$ then $(f \circ h)^* = f^* \circ Sh$.

So suppose that $F : \mathbb{I} \to \mathbb{C}$ is a diagram and $c, \{f_a\}$ is a limit of this diagram. Consider an arbitary cone $d, \{g_a\}$ over the diagram $T \circ F$. Explicitly $g_a : d \to TFa$ and if $h : a \to b$ in $\mathbb{I}$ then $g_b = TFh \circ g_a$.

The adjunction gives us for each g_a a corresponding morphism $g_a^* : Sd \to Fa$, and by N1 above $g_b^* = (TFh \circ g_a)^* = Fh \circ g_a^*$ so that $Sd, \{g_a^*\}$ gives us a cone over F. By the definition of limit there is a unique morphism of cones from this cone to the limit cone $c, \{f_a\}$, or more explicitly there is a unique morphism $h : Sd \to c$ such that $g_a^* = f_a \circ h$ for all a.

Now $h = i^*$ for some $i : d \to T(c)$, so by N1 again $g_a^* = f_a \circ i^* = (Tf_a \circ i)^*$ and hence $g_a = Tf_a \circ i$ so that i gives a cone morphism from $d, \{g_a\}$ to $Tc, \{Tf_a\}$. If j is any such cone morphism then by yet another appeal to N1 we have

$$g_a^* = (Tf_a \circ j)^* = f_a \circ j^*$$

for all a, so $i^* = h = j^*$ and thus $i = j$. We have showed that $Tc, \{Tf_a\}$ is a limit as required.

So what is N2 good for? It can be used to show that S preserves *colimits* which are defined as follows: a *cocone* over a diagram $G : \mathbb{I} \to \mathbb{D}$ is given by $d, \{g_b : Gb \to d\}$ such that if $h : a \to b$ then $g_b \circ Gh = g_a$. A *morphism* from $d_1, \{g_b^1\}$ to $d_2, \{g_b^2\}$ is $g : d_1 \to d_2$ such that $g \circ g_b^1 = g_b^2$ for all b, and a *colimit* is an initial cocone.

Just to make life more confusing: in algebra and topology limits as defined here are often called "inverse limits" and colimits are "direct limits".

Irritating point: exact sequences of rings do not behave so nicely because my convention on HMs implies that a HM $0 \to R$ only exists when $R = 0$. Since I want to use exact sequences in a discussion of inverse limits and completions of rings I will adopt the following procedure: prove results about inverse limits and completions of abelian groups (a context where exact sequences behave well) and then remember the ring structure at the end.

Convention: until further notice all groups are abelian and are written additively (group operation is $+$, identity is 0, inverse of g is $-g$)

Inverse limits of groups: we have (abelian!) groups G_n for $n \in \mathbb{N}$ and maps π_{nm} for $m \leq n$ satisfying $\pi_{mm} = id$, and $\pi_{ca} = \pi_{ba} \circ \pi_{cb}$ for $a \leq b \leq c$. This is a special case of the kind of diagram considered above.

Theorem: if we define $\lim\limits_{\leftarrow} G_n$ to be the set of all sequences $(g_n : n \in \mathbb{N})$ such that $\pi_{nm}(g_n) = g_m$ for $m \leq n$ (eqivalently $\pi_{n+1n}(g_{n+1}) = g_n$ for all n) then

(1) $\lim\limits_{\leftarrow} G_n$ is a group under coordinatewise addition.
(2) The maps $f_i : (g_n) \mapsto g_i$ commute with the π_{mn} so we have a cone.
(3) The cone we just described is a limit of the diagram of G's and π's.

Proof: easy!

Note that (0) is the identity. We say that the system is *surjective* when all the π_{nm} (or equivalently all the π_{n+1n}) are surjective. Example: the inverse limit system defining $\mathbb{Z}_p$ on today's HW.

Of course we want to make the collection of inverse limit systems into a category. Not too surprisingly: a morphism from the system with groups G_n and maps π^G_{nm} to the system with groups H_n and maps π^H_{nm} consists of group HMs $f_n : G_n \to H_n$ such that $f_m \circ \pi^G_{nm} = \pi^H_{nm} \circ f_n$ for all $m \leq n$ (equivalent: $f_n \circ \pi^G_{n+1n} = \pi^H_{n+1n} \circ f_{n+1}$ for all n)

Peter Lumsdaine points out that if we think of the inverse limit systems as functors (defined on a category whose objects are the elements of $\mathbb{N}$, with a unique morphism from n to m iff $m \leq n$) then these morphisms of inverse limit systems are precisely the natural transformations.

Now routine to check that the inverse limit construction gives us a functor from inverse limit systems to groups. What does it preserve? This brings us to something I've been putting off.

Homological algebra: Fix ring R. A chain complex of R-modules is a sequence of R-modules C_n (n may run through any contiguous range of integers) together with HMs $\partial_n : C_{n+1} \to C_n$ such that $\partial_n \circ \partial_{n+1} = 0$ for all n where this makes sense.

Remark: this is a weakening of the concept of exact sequence. Applying some reasonable functor to an exact sequence will typically give a chain complex.

Define "homology modules" to measure the failure of exactness: formally $Z_n(C) = ker(\partial_{n-1})$ and $B_n(C) = im(\partial_n)$. Clearly $B_n(C) \leq Z_n(C) \leq C_n$, we define the nth homology module $H_n(C) = Z_n(C)/B_n(C)$.

Cultural note: This comes from topology where the ∂_n are "boundary maps". Often we speak of B_n as consisting of "boundaries" and Z_n as consisting of "cycles".

Remark: the homology of a sequence $0 \to A \to B \to 0$ at A is just the kernel of the map $A \to B$, and the homology at B is just the cokernel.

A morphism of chain complexes is just a family of HMs which commute with the boundary maps. Explicitly if we have C_n and D_n then the morphism f is a collection of maps $f_n : C_n \to D_n$ such that $f_n \circ \partial_n^C = \partial_n^D \circ f_{n+1}$.

Routinely: f_n maps $Z_n(C)$ to $Z_n(D)$ and $B_n(C)$ to $B_n(D)$, and so induces a map $H_n(f) : H_n(C) \to H_n(D)$ by $z + B_n(C) \mapsto f_n(z) + B_n(D)$. H_n is a functor from chain complexes of R-modules to R-modules.

THIS BIT TO BE FILLED IN.