CA LECTURE 19

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Let k be a field and M a k-module (that is a vector space over k). Then M is A'ian iff M is N'ian iff M is finite dimensional iff M is fg. The proof is easy linear algebra.

Definition: if R is a ring then a *chain of length* n in R is $P_0 \subsetneq \ldots \subsetneq P_n$ with the P_i prime. The *dimension of* R is the sup of the lengths of the chains of prime ideals, or ∞ if there exist chains of unbounded length.

Easily $dim(R) = 0$ iff every prime ideal is maximal.

Recall that if I is an ideal then I^n is the ideal generated by all products $a_1 \ldots a_n$ with $a_i \in I$. We say that I is *nilpotent* iff $I^n = 0$ for some n.

Easy remark: in a N'ian ring the nilradical is nilpotent.

Theorem: if R is A'ian then the nilradical is nilpotent.

Proof: Let N be the nilradical and consider the decreasing chain of ideals N^k. Suppose N is not nilpotent then $N^k = I \neq 0$ for all large k.

Let X be the set of ideals J with $IJ \neq 0$, then X is not empty because R (or I or N) is in X. let J be minimal in X.

There is $c \in J$ with $cI = (c)I \neq 0$, and of course $(c) \subseteq J$ so that by minimality $(c) = J$. Now $(cI)I = cI^2 = cI \neq 0$ and $cI \subseteq J$, so $cI = J$ by minimality again. Since $c \in J$ we have $c = cd$ for some $d \in I$. d is nilpotent and so $1 - d$ is a unit, hence $c = 0$ and we have a contradiction.

Lemma: Let R be a ring such that the zero ideal is a finite product of (not necessarily distinct) maximal ideals $M_1 \ldots M_n$. Then R is N'ian iff R is A'ian.

Proof: Let $I_0 = R$ and $I_j = M_1 \ldots M_j$ for $j > 0$. Then as we remarked at the end of last time:

(1) The ideals K of R such that $I_{j+1} \subseteq I_j$ are in an inclusion-preserving bijection with the R-submodules of I_j/I_{j+1}.
(2) I_j/I_{j+1} can be seen as an R-module or as an R/M_{j+1}-module.
(3) The R-submodules of I_j/I_{j+1} are precisely the subspaces when we consider it as a VS over the field R/M_{j+1}.
(4) In particular I_j/I_{j+1} is an Artinian R-module iff it is a FD VS over R/M_{j+1} iff it is a Noetherian R-module.

Now suppose that R is a N'ian ring. By standard properties of N'ian modules, each ideal I_j and each quotient I_j/I_{j+1} is a N'ian R-module. So each I_j/I_{j+1} is an A'ian R-module. Now we argue by backwards induction that each I_j is an *A*'ian R-module, using the fact from last time that for a module to be A'ian it is sufficient to have an A'ian submodule with an A'ian quotient.

Theorem : TFAE for a ring R

(1) R is A'ian.
(2) R is N'ian of dimension zero.

Proof: First let R be A'ian. The nilradical N is the intersection of all the prime ideals, so (by work last time) $N = M_1 \cap \dots M_n$ where the M_i are the finitely many maximal ideals. Fix k with $N^k = 0$, then

$$M_1^k \dots M_n^k = (M_1 \dots M_n)^k \subseteq (M_1 \cap \dots M_n)^k = N^k = 0.$$

So 0 is a product of maximal ideals and thus R is N'ian. We already saw that R has dimension zero.

Conversely let R be N'ian of dimension zero. Since R is N'ian all ideals are decomposable in particular we may fix an irredundant decomposition $0 = Q_1 \cap \dots Q_n$ where the Q_i are P_i-primary. Taking radicals the nilradical N is

$$\sqrt{0} = \sqrt{Q_1 \cap \dots Q_n} = \sqrt{Q_1} \cap \dots \sqrt{Q_n} = P_1 \cap \dots P_n.$$

Since R has dimension zero the P_i are maximal, and since R is N'ian the nilradical N is nilpotent. Now we argue exactky as before that 0 is a product of maximal ideals and thus that R is Artinian.

Now we make our first serious use of Nakayama: suppose that R is a N'ian local ring with maximal ideal M. Note that the Jacobson radical of R is M and that all ideals of R are fg as R-modules. Consider the decreasing chain of ideals M^n. It may stabilise or not. If $M^n = M^{n+1}$ then applying Nakayama with M as the ideal and M^n as the fg R-module we see $M^n = 0$, in which case arguing as in the last theorem R is A'ian. Otherwise the M^n form an infinite strictly decreasing chain.

Defn: Ideals I and J in a ring R are *comaximal* iff $I + J = R$.

Lemma: If I and J are comaximal then $I \cap J = IJ$.

Proof: As usual $IJ \subseteq I \cap J$. let $1 = a + b$ for $a \in I$, $b \in J$ and let $c \in I \cap J$; then $c = ac + cb \in IJ$.

Lemma: If $\sqrt{I}$ and $\sqrt{J}$ are comaximal then so are I and J.

Proof: Otherwise let P be prime with $I + J \subseteq P$. Then $I \subseteq P$ so $\sqrt{I} \subseteq \sqrt{P} = P$, similarly $J \subseteq P$, and $I + J \subseteq P \neq R$. Contradiction!