COMMUTATIVE ALGEBRA HW 5 SOLUTIONS

JC

Due in class Wed 14 September.

(1) Let R be a UFD. Show that $R[x]$ has infinitely many pairwise non-associate irreducibles.

We imitate Euclid's argument to produce an infinite sequence of distinct monic irreducibles (note that a monic irreducible has degree at least one and distinct monic polynomials are not associates).

x is monic irreducible. Let $f_1, \dots f_n$ be distinct monic irreducibles. Then $f_1 \dots f_n + 1$ is monic of positive degree. In its prime factorisation each factor has leading coefficient a unit so we may as well assume that each prime factor is monic. Clearly none of the prime factors appear among the f_i.

Addendum: The original (broken) version of Q1 asked to show that a UFD with at least one irreducible has infinitely many irreducibles. Whether this is true depends on whether we mean "infinitely many distinct irreducibles" (true as shown by Spas) or "infinitely many non-associate irreducibles" (false as shown by Yimu). The true version is not sufficient to do Q2.

Yimu's example: let R be the subring of $\mathbb{Q}$ consisting of rationals a/b with b odd. A little thought shows that R is a local PID (hence UFD) with maximal ideal (2), 2 is the only prime up to associates. But of course there are infinitely many units so actually 2 has infinitely many associates. Cultural note: this is an example of a DVR (= Discrete Valuation Ring), a class of rings which is important in number theory and geometry.

Spas' proof: If the UFD R has infinitely many units we are done. So suppose we have only finitely many units. Suppose for a contradiction that there are finitely many primes $p_1, \dots p_n$. Since we are in a UFD the elements $p_1^t p_2 \dots p_m + 1$ (for $t > 0$) are all distinct so at least one is a nonunit, now we get the usual contradiction a la Euclid.

(2) Let K be a field, let $n \geq 1$ and let $K(x_1, \dots x_n)$ be the field of fractions of the polynomial ring $K[x_1, \dots x_n]$. Show that

$K(x_1, \dots x_n)$ is not ring-finite/K. Hint: $K[x_1, \dots x_{n-1}]$ is a UFD.

Suppose for a contradiction that $K(x_1, \dots x_n) = K[r_1, \dots r_k]$ where $r_i = f_i/g_i$ for $f_i, g_i \in K[x_1, \dots x_n]$ with $g_i \neq 0$. Let h be an irreducible of $K[x_1, \dots x_n]$ which does not divide any g_i. We may express $1/h$ as a polynomial in the r_i with coefficients from K, and clearing fractions we get an equation in $K[x_1, \dots x_n]$ of the shape $g_1^{n_1} \dots g_k{}^{n_k} = hF$, contradicting our choice of h.

(3) We proved in class that if $A \leq B \leq C$ are rings with A Noetherian and C both module-finite/B and ring-finite/A then B is ring-finite/A, Show that this is false if we drop the hypothesis that A is Noetherian.

Let $R = \mathbb{Z}[x_0, x_1, \dots]$. This polynomial ring in infinitely many variables is our prototypical non-Noetherian ring.

Let $I = (x_0^2)_R$ and let $C = R/I$. Routinely if $y_i = x_i + I$ then $C = \mathbb{Z}[y_0, y_1, \dots]$ (I am cheating a little bit here by identifying $n \in \mathbb{Z}$ with its coset $n+I$ in the quotient ring C, since $n \mapsto n+I$ is 1-1 it gives an isomorphic copy of $\mathbb{Z}$ inside C).

Let $A = \mathbb{Z}[y_1, y_2, \dots]$ and let $B = A[y_0 y_1, y_0 y_2, \dots]$. Clearly $C = A[y_0]$ and since $y_0^2 = 0$ in fact $C = (1, y_0)_A$ so that C is module-finite over A and hence *a fortiori* over B.

Suppose for a contradiction that B is ring-finite over A, say $B = A[b_1, \dots b_l]$. Since each b_l is in $A[y_0 y_1, \dots y_0 y_m]$ for m sufficiently large we will then have that $B = A[y_0 y_1, \dots y_0 y_m]$ for some large m. In particular $y_0 y_{m+1} \in A[y_0 y_1, \dots y_0 y_m]$.

Going back to R it follows that there is some polynomial F with integer coefficients and some $n \geq m$ such that $x_0 x_{m+1} - F(x_1, \dots x_n, x_0 x_1, \dots x_0 x_m)$ is in I, that is WLOG we have a polynomial identity for some polynomial with integer coefficients

$$x_0 x_{m+1} = F(x_1, \dots x_n, x_0 x_1, \dots x_0 x_m) + G(x_0, \dots x_n) x_0^2.$$

But this is impossible: two polynomials in $\mathbb{Z}[x_0, \dots x_n]$ are equal iff each monomial $x_0^{a_0} \dots x_n^{a_n}$ has the same coefficient and the monomial $x_0 x_{m+1}$ has coefficient zero on the RHS.

(4) Let X be a topological space. We say that $A \subseteq X$ is *closed* iff its complement $X \setminus A$ is open. Given a ring R, identify the prime ideals P such that $\{P\}$ is closed in $Spec(R)$.

Using the definitions, $\{P\}$ is closed iff $Spec(R) \setminus \{P\}$ is open iff for every prime $Q \neq P$ there is a such that $Q \in O_a \subseteq Spec(R) \setminus \{P\}$ iff for every prime $Q \neq P$ there is $a \in P \setminus Q$ iff for every prime $Q \neq P$ we have $P \not\subseteq Q$.

If P is maximal then clearly this holds. If P is not maximal then $P \subsetneq Q$ for some maximal (hence prime) Q. So in conclusion $\{P\}$ is closed iff P is maximal.