# COMMUTATIVE ALGEBRA HW 23 SOLNS

JC

The axioms:

(1) $A$ is a subset of $cl(A)$.
(2) If $A$ is a subset of $cl(B)$ then $cl(A)$ is a subset of $cl(B)$.
(3) If $x$ is in $cl(A)$ then there is a finite subset $A_0$ of $A$ such that $x$ is in $cl(A_0)$.
(4) If $x$ is in $cl(A \cup \{y\})$ and $x$ is not in $cl(A)$ then $y$ is in $cl(A \cup \{x\})$.

(1) If $A$ is a subset of $B$ then $cl(A)$ is a subset of $cl(B)$.

Proof: $A \subseteq B \subseteq cl(B)$ by Axiom 1, so $cl(A) \subseteq cl(B)$ by Axiom 2.

(2) $cl(cl(A)) = cl(A)$.

By Axiom 1, $cl(A) \subseteq cl(cl(A))$. Also since $cl(A) \subseteq cl(A)$, by Axiom 2 we have $cl(cl(A)) \subseteq cl(A)$.

(3) A set $A$ is independent if $a$ is not in $cl(A \setminus \{a\})$ for all $a \in A$. An independent set $B$ is a basis if in addition $cl(B) = X$. [Note that in the VS example these concepts have the usual meanings]

Trivially (by what you proved above) a subset of an independent set is independent.

Handy lemma: if $A$ is independent and $a \notin cl(A)$ then $A \cup \{a\}$ is independent (a lot of you assumed this is true without providing a proof, just showing that you have sound intuition!)

Proof of lemma: Let $B = A \cup \{a\}$. Then $a \notin cl(B \setminus \{a\})$ by assumption. Let $b \in A$, and note that $B \setminus \{b\} = (A \setminus \{b\}) \cup \{a\}$. Now $b \notin cl(A \setminus \{b\})$ as $A$ is independent, and so by Axiom 4 if $b \in cl(B \setminus \{b\})$ then $a \in cl(A \setminus \{a\} \cup \{a\}) = cl(A)$, contradiction.

Show that an independent set is a basis iff it is a maximal element in the poset of independent sets ordered by inclusion.

Proof: If $B$ is a basis then $b \in cl(B)$ for all $b$, so $B \cup \{b\}$ is not independent for any $b \notin B$.

Conversely if $B$ is maximal independent then we must have $cl(B) = X$, for otherwise we can find $b \notin cl(B)$ and then by the Lemma above $B \cup \{b\}$ is independent.

(4) Show that the union of a chain of independent sets is independent, and use ZL to conclude that any independent set can be extended to a basis.

Let $\mathcal{C}$ be a chain of independent sets and let $A = \cup\mathcal{C}$. Let $a \in A$ and suppose for a contradiction that $a \in cl(A \setminus \{a\})$. By Axiom 3 there is a finite set $A_0 \subseteq A \setminus \{a\}$ with $a \in cl(A_0)$. Now since $\mathcal{C}$ is a chain any finite subset of $A$ is contained in some element of $\mathcal{C}$, in particular we may find $B \in \mathcal{C}$ with $A_0 \cup \{a\} \subseteq B$. But now we get $a \in cl(A_0) \subseteq cl(B \setminus \{a\})$, contardiction as $B$ is independent.

(5) Show that if B is a basis, then for any element $x$ of $X$ there is a finite subset $C$ of $B$ such that $x$ is in $cl(C)$, but $x$ is not in $cl(C')$ for any proper subset $C'$ of $C$.

Proof: By Axiom 3 there is finite $C \subseteq B$ with $x \in cl(C)$. Choose such a $C$ with $|C|$ minimal.

Show further that if $c$ is in $C$, then the set obtained from $B$ by replacing $c$ by $x$ is also a basis.

Proof: Let $B_1 = B \setminus \{c\}$ and $B_2 = B_1 \cup \{x\}$.

We observe that by construction $x \in cl(C)$ but $x \notin cl(C \setminus \{c\})$, so by Axiom 4 we have $c \in cl(C \setminus \{c\} \cup \{x\})$, so a fortiori we have $c \in cl(B_1 \cup \{x\})$. It follows from this that $x \notin cl(B_1)$, because if $x \in cl(B_1)$ then $c \in cl(B_1)$, contradicting the independence of $B$.

Since $B_1$ is independent, it follows from the Lemma above that $B_2 = B_1 \cup \{x\}$ is independent. We claim it is also a basis. To see this note that $c \in cl(B_2)$, and also $B_1 \subseteq B_2 \subseteq cl(B_2)$, so that $B \subseteq cl(B_2)$ and hence $X = cl(B) \subseteq cl(B_2) \subseteq X$.

(6) Show if $\{x_1, \ldots x_n\}$ is an independent set and $B$ is a basis then $B$ has at least $n$ elements.

Start by noting that $\{b\}$ is independent iff $b$ is in some basis iff $b \notin cl(\emptyset)$,

We show by induction on $n$ that there exist $y_1, \ldots y_n$ distinct elements of $B$ such that $B \setminus \{y_1, \ldots y_n\} \cup \{x_1, \ldots x_n\}$. The case $n = 1$ is handled by the previous item.

Induction step: Let $m = n + 1$, so that by induction we find distinct $y_1, \ldots y_n$ in $B$ such that $B' = B \setminus \{y_1, \ldots y_n\} \cup \{x_1, \ldots x_n\}$ is a basis. Notice that it's not ruled out that some of the $y_i$ for $1 \leq i \leq n$ are in $B'$ but this is only possible if they are among $\{x_1, \ldots x_n\}$.

Choose a finite $C \subseteq B'$ with $x_m \in cl(C)$ and $|C|$ minimal. Now since $\{x_1, \ldots x_n, x_m\}$ is independent we see that $x_m \notin cl(\{x_1, \ldots x_n\})$, and so $C \nsubseteq \{x_1, \ldots x_n\}$. Choose $c \in C \setminus \{x_1, \ldots x_n\}$ and let $y_m = c$, this clearly works.

(7) Show that if a basis of size $n$ exists, all bases have size $n$.

Let $B$ be a basis of size $n$ and let $B'$ be an arbitrary basis. Since $B'$ is independent it must have size at most $n$, otherwise it would contain an independent set of size $n+1$ contradicting the previous item. But sinec $B$ is independent and $B'$ is a basis $B'$ has size at least $n$.

Note: The phrasing may sound contorted but notice that a priori $B'$ might be infinite.