COMMUTATIVE ALGEBRA HW 17 SOLNS

JC

(1) True or false?
  (a) Any nonempty dense subset of an irreducible space is irreducible.
  (I should have said "in the subspace topology"). True: Let D be dense in X. We claim that any two nonempty open subsets intersect. For any two such sets have form $U \cap D$ and $V \cap D$ for U, V nonempty open sets in X, then $U \cap V$ is nonempty (as X is irreducible) and open so that $U \cap V \cap D \neq \emptyset$.
  (b) The image of an open set under a continuous function is always open
  False. Consider for examaple the map $(x, y) \mapsto (x, 0)$ on $\mathbb{R}^2$ with the usual metric topology.
  (c) The set of maximal ideals is always closed in $Spec(R)$.
  False. Let $R = \mathbb{Z}$ so that the maximal ideals are (p) for p prime (that is all the nonzero prime ideals). We claim that this set is not closed. Suppose it is, then the singleton of (0) is open and so there is a such that $(0) \in O_a$ and $(p) \notin O_a$ for all p. Now $a \neq 0$ so a has only finitely many prime factors, hence by Euclid there is p such that $(p) \in O_a$.

(2) Let p be prime , let $R_n = \mathbb{Z}/p^n\mathbb{Z}$ and (as in class) define maps π_{nm} for $m \leq n$ by $\pi_{nm} : a + p^n\mathbb{Z} \mapsto a + p^m\mathbb{Z}$. These are ring HMs and easily $\pi_{nn} = id$ and $a \leq b \leq c$ implies $\pi_{ca} = \pi_{ba} \circ \pi_{cb}$.

Form an "inverse limit ring" $\mathbb{Z}_p$ in the obvious way, namely elements are sequences $(r_0, r_1, r_2, \ldots)$ such that $\pi_{n+1n} r_{n+1} = r_n$ for all n or equivalently $\pi_{nm} r_n = r_m$ for $m \leq n$. The ring operations are defined coordinatewise.
  (a) Show $\mathbb{Z}_p$ contains an isomorphic copy of $\mathbb{Z}$ (which we will identify with $\mathbb{Z}$ in what follows).
  Consider the map which takes integer n to the sequence $(n + p^i\mathbb{Z})_i$. Routinely it's a HM with zero kernel, hence injective.
  (b) Show that -1 has a square root in $\mathbb{Z}_5$. How many are there?

We are seeking a_n such that a_n^2 is congruent to $-1 \bmod 5^n$, and a_{n+1} is congruent to $a_n \bmod 5^n$. We start by choosing $a_1 = 2$ and show by induction that this gives an essentially unique solution for every $n \geq 1$. Suppose a_n has been determined, and that $a_n^2 + 1 = 5^n M$ for some M. We seek a_{n+1} in the form $a_n + 5^n x$ for some x. Necessarily $(a_n^2 + 2a_n 5^n x + 5^{2n} x^2) + 1$ is a multiple of 5^{n+1}, that is $5^n M + 2a_n 5^n x$ is a multiple of 5^{n+1}, or equivalently $M + 2a_n x$ is a multiple of 5. This equation can be solved for x because a_n is a not a multiple of 5, and what is more x is unique mod 5. So we can find a suitable a_{n+1} which is unique mod 5^{n+1}.

For the last part: either observe that $a_1 = 3$ also works and nothing else does, or prove that $\mathbb{Z}_5$ is an ID so that a nonzero polynomial of degree n has at most n roots.