COMMUTATIVE ALGEBRA HW 12

JC

Due in class Wed 5 October.

(1) Let I be an ideal which has a primary decomposition. Show that if I is radical then I has no embedded prime ideals.

Let $I = Q_1 \cap \dots Q_m$ irredundantly. $I = \sqrt{I} = \sqrt{Q_1} \cap \dots \sqrt{Q_m} = P_1 \cap \dots P_m$. The P_i are primary (indeed prime) so they must form another irredundant decomposition, otherwise we could delete redundancies and get a new primary decomposition involving a different set of primes (which is impossible because as we saw I determines the set of primes in a primary decomposition).

Now irredundancy implies that no P_i contains another P_j that is all the primes are minimal.

(2) Let $R = \mathbb{Z}[x]$. Let $M = (2, x)_R$ and $I = (4, x)_R$. Show that
   (a) M is maximal. Hint: what is R/M?
   (b) I is primary.
   (c) $\sqrt{I} = M$.
   (d) $I \neq M^n$ for all $n > 0$.

M is the ideal of polynomials with constant term a multiple of 2, and I is the ideal of polynomials with constant term a multiple of 4.

M is maximal because $R/M \simeq \mathbb{Z}/2\mathbb{Z}$ is an ID, I is primary because in $R/I \simeq \mathbb{Z}/4\mathbb{Z}$ the ZD's 0 and 2 are both nilpotent. $M = \sqrt{I}$ is obvious. All polynomials in M^n have constant term a multiple of 2^n so the only possibility is that $I = M^2 = (4, 2x, x^2)$. But in M^2 all polynomials have the coefficient of x even.