## COMMUTATIVE ALGEBRA HW 12

JC

Due in class Wed 5 October.

(1) Let I be an ideal which has a primary decomposition. Show that if I is radical then I has no embedded prime ideals.

Let  $I = Q_1 \cap \ldots Q_m$  irredundantly.  $I = \sqrt{I} = \sqrt{Q_1} \cap \ldots \sqrt{Q_m} = P_1 \cap \ldots P_m$ . The  $P_i$  are primary (indeed prime) so they must form another irredundant decomposition, otherwise we could delete redundancies and get a new primary decomposition involving a different set of primes (which is impossible because as we saw I determines the set of primes in a primary decomposition).

Now irredundancy implies that no  $P_i$  contains another  $P_j$  that is all the primes are minimal.

- (2) Let  $R = \mathbb{Z}[x]$ . Let  $M = (2, x)_R$  and  $I = (4, x)_R$ . Show that
  - (a) M is maximal. Hint: what is R/M?
  - (b) I is primary.
  - (c)  $\sqrt{I} = M$ .
  - (d)  $I \neq M^n$  for all n > 0.

M is the ideal of polynomials with constant term a multiple of 2, and I is the ideal of polynomials with constant term a multiple of 4.

M is maximal because  $R/M \simeq \mathbb{Z}/2\mathbb{Z}$  is an ID, I is primary because in  $R/I \simeq \mathbb{Z}/4\mathbb{Z}$  the ZD's 0 and 2 are both nilpotent.  $M = \sqrt{I}$  is obvious. All polynomials in  $M^n$  have constant term a multiple of  $2^n$  so the only possibility is that  $I = M^2 =$  $(4, 2x, x^2)$ . But in  $M^2$  all polynomials have the coefficient of xeven.