COMMUTATIVE ALGEBRA HW 11

JC

Due in class Mon 3 October.

(1) Let M be an R-module and define an $R[x]$-module $M[x]$ as follows: the elements are all finite sums $\sum_{i=0}^n m_i x^i$ where $n \in \mathbb{N}$ and $m_i \in M$, with the obvious operations for $+$ and scalar multiplication. Show that $M[x]$ is isomorphic to $R[x] \otimes_R M$ as an $R[x]$-module. Hint: what is the obvious R-bilinear map from $R[x] \times M$ to $M[x]$?

Consider the map $\gamma : (\sum_i r_i x^i, m) \mapsto \sum_i r_i m x^i$. This is obviously R-bilinear (we note in passing that $M[x]$ is a natural example of an $(R, R[x])$-bimodule).

We claim it has the usual "universal property", that is every R-bilinear map is $\delta \circ \gamma$ for a unique R-linear δ. So let $\phi : R[x] \times M \to C$ be R-bilinear and observe that

$$\phi(\sum_i r_i x^i, m) = \sum_i \phi(x^i, r_i m).$$

Now if we define $\delta : M[x] \to C$ by

$$\delta : \sum_i m_i x^i \mapsto \sum_i \phi(x^i, m_i)$$

then δ is linear (bevause ϕ is linear in its second argument) and

$$\delta(\gamma(\sum_i r_i x^i, m)) = \delta \sum_i r_i m x^i = \sum_i \phi(x^i, r_i m) = \phi(\sum_i r_i x^i, m).$$

Finally if δ^* is linear with $\delta^* \circ \gamma = \phi$ then

$$\delta^*(m x^j) = \delta^*(\gamma(x^j, m)) = \phi(x^j, m),$$

so by linearity $\delta = \delta^*$.

This establishes that there is an IM of R-modules between $R[x] \otimes M$ and $M[x]$ such that $f \otimes m \mapsto fm$. Now it is routine to check (using the definition of the $R[x]$-scalar multi[plication on the tensor product) that this is automatically also an IM of $R[x]$-modules.

(2) Show that if M is a Noetherian R-module then $M[x]$ is a Noetherian $R[x]$-module. Hint: what is this saying when $M = R$?

The case $M = R$ is the Hilbert Basissatz. We can imitate the proof of the Basissatz, using the fact that M^n and every submodule of M^n are Noetherian R-modules for all n (this follows by a similar induction to the one used to show that if R is a Noetherian ring then R^n is a Noetherian R-module). More details can be provided on request if you don't see how this will go.

(3) Let I be an ideal of R and let $S = 1 + I$, that is to say $\{1 + a : a \in I\}$. Show that
   (a) S is a multiplicatively closed subset of R.
   Easy!
   (b) Let $S^{-1}I$ be the extension of I in $S^{-1}R$, that is to say $S^{-1}I = \{a/s : a \in I, s \in S\}$. Show that $S^{-1}I$ is contained in the Jacobson radical of $S^{-1}R$.
   It is enough to show that every element of $1 + S^{-1}I$ is a unit, as $S^{-1}R$ is an ideal. If $a \in I$ and $s \in S$ then $1 + a/s = (a + s)/s$ where easily $a + s \in 1 + I = S$, so that $1 + a/s$ is a unit with inverse $s/(a + s)$.