COMMUTATIVE ALGEBRA FINAL

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You may work on this exam in any continuous 24 hour period between now and midnight on December 18. Collaboration is not permitted. You may refer to the class notes and text, and to any other books or papers (please make a note of works used). Please consult me by email if you have a problem (the 24 hour clock stops ticking until I reply).

You should attempt exactly seven questions.

(1) True or false:
   (a) The product of two IDs is an ID.
   (b) The product of two Noetherian rings is Noetherian.
   (c) The product of two Artinian rings is Artinian.
   (d) The product of two local rings is local.

   Hint: you should prove a theorem identifying the ideals of $R \times S$ before you do the later parts.

(2) Let R be a ring such that every ideal not contained in the nilradical contains a nonzero *idempotent element* (that is to say an element e with $e^2 = e$). Prove that the nilradical equals the Jacobson radical.

(3) Let R be a ring, and I an ideal contained in the Jacobson radical of R. Let M and N be R-modules where N is fg, and let $\phi : M \to N$ be an R-module HM. Show that
   (a) ϕ induces a HM $\phi^* : M/IM \to N/IN$.
   (b) If ϕ^* is surjective then ϕ is surjective.

(4) Let $\phi : R \to S$ be a ring HM and let N be an S-module. Make N into an R-module by defining $rn = \phi(r)n$, and then form the S-module $N^* = S \otimes_R N$. Define a map $g : N \to N^*$ by $g : n \mapsto 1_S \otimes n$.
   (a) Show that there is a unique map $p : N^* \to N$ such that $p : s \otimes n \mapsto sn$.
   (b) Show that g is injective and that N^* is the internal direct sum of $im(g)$ and $ker(p)$.

(5) (Local properties, see Ch 3 of A and M) Let R be a ring. Show that if for every prime ideal P the ring R_P has no nonzero nilpotent elements, then R has no nonzero nilpotent elements.

(6) Let R be a ring. Recall that if $a \in R$ then $R_a = S_a^{-1}R$ where $S_a = \{1, a, a^2, \ldots\}$, and $O_a = \{P \in Spec(R) : a \notin P\}$. Show that if $O_a = O_b$ then $R_a \simeq R_b$.

(7) Let k be a field and let $R = k[x, y, z]$ be a polynomial ring in three variables over k. Let $P_1 = (x, y)$, $P_2 = (x, z)$, $M = (x, y, z)$, $A = P_1P_2$.
   (a) Show P_1 and P_2 are prime, M is maximal.
   (b) Show $A = P_1 \cap P_2 \cap M^2$ is an irredundant primary decomposition of A.
   (c) Identify the minimal and embedded primes.

(8) True or false: if M is an Artinian R-module then $R/Ann(M)$ is an Artinian ring?

(9) Let G be a finite group of auotmorphisms of the ring R and let $R^G = \{r : \forall \sigma \in G\ \sigma(r) = r\}$.
   (a) Show that R^G is a subring of R.
   (b) By considering the polynomial $\prod_{\sigma \in G}(x - \sigma(r))$ for a typical $r \in R$, or otherwise, show R is integral over the subring R^G.
   (c) Let S be a multiplicatively closed set such that $\forall \sigma \in G\ \forall s \in S\ \sigma(s) \in S$. Let $S^G = S \cap R^G$. Find an action of the group G on $S^{-1}R$ so that $(S^{-1}R)^G \simeq (S^G)^{-1}R^G$.

(10) Let M be an R-module with submodules N_1 and N_2. Show that if M/N_1 and M/N_2 are Noetherian then so is $M/(N_1 \cap N_2)$.

(11) Let R be a ring such that R_P is Noetherian for every prime ideal P. Is R necessarily Noetherian?

(12) Recall that a topological space is discrete iff every set is open (equivalently all points are open). Let R be a Noetherian ring. Show that TFAE:
   (a) R is Artinian.
   (b) $Spec(R)$ is discrete and finite.
   (c) $Spec(R)$ is discrete.

   Hint: For any ring R, $Spec(R)$ is compact; to see this just note that if a family of basic open sets O_a for $a \in A$ has union $Spec(R)$, then $R = (A)$ and so $R = (A_0)$ for a finite $A_0 \subseteq A$.