ALGEBRA HOMEWORK SET 7

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Due by class time on Wednesday 2 November. Homework must be typeset and submitted by email as a PDF file.

(1) Recall that a subset S of a ring R is multiplicatively closed (MC) if $1 \in S$ and S is closed under multiplication. Let $S \subseteq R$ be MC and define a binary relation $\sim$ on $R \times S$ as follows: $(r, s) \sim (r', s')$ iff there is $t \in S$ such that $t(rs' - r's) = 0$. Prove that:
   (a) $\sim$ is an equivalence relation.
   (b) Defining $+$ and $\times$ as in the definition of field of fractions makes the set of $\sim$-classes into a ring (which we write RS^{-1}). Just check the operations are well-defined, it is then clear that the ring axioms are satisfied.

(2) Let G be a torsion-free $\mathbb{Z}$-module (abelian group) of rank 1 and let P be the set of prime numbers.
   (a) Prove that G is isomorphic to a subgroup of $(\mathbb{Q}, +)$.
   (b) Let $G \leq \mathbb{Q}$. For each nonzero $a \in G$, let $n_a : P \to \mathbb{N} \cup \{\infty\}$ be defined as follows:

$$n_a(p) = \sup\{n : a/p^n \in G\}.$$

   Prove that if a and b are both nonzero then $n_a(p) = \infty \iff n_b(p) = \infty$, and $\{p : n_a(p) \neq n_b(p)\}$ is finite.

(3) Let A be an $n \times n$ integer matrix and let G_A be the subgroup of $\mathbb{Z}^n$ generated by the columns of A. Prove that $\mathbb{Z}^n/G_A$ is finite iff $det(A) \neq 0$, and that in this case $\mathbb{Z}^n/G_A$ has order $|det(A)|$.

(4) Prove that the intersection of any nonempty chain of prime ideals is prime.

(5) Let R be a PID and let N be a free R-module on a countably infinite set of generators (for example the set of all functions from $\mathbb{N}$ to R which are zero on a cofinite set). Prove that every submodule of N is free.