ALGEBRA HOMEWORK SET 3 SOLUTIONS

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(1) Prove that a finite group G is solvable if and only if its composition factors are cyclic groups of prime order.

Solvability is inherited by quotients and subgroups, so the factors of a composition series in a finite solvable group will be finite groups which are both solvable and simple, hence are cyclic of prime order. Conversely if the composition factors are cyclic of prime order, then any composition series is a subnormal series with abelian quotients, so that G is solvable.

(2) Compute the derived series of S_4. Hint: Keep in mind that $[G, G]$ is the smallest normal subgroup of G with an abelian quotient, this can save you computing too many commutators. Is the derived series of S_4 a composition series? Find the composition factors of S_4. Is S_4 nilpotent?

Since S_4/A_4 is abelian, the derived subgroup of S_4 is contained in A_4. Also $(12)(13)(12)(13) = (123)$, so that (normality!) every 3-cycle is a commutator. Finally $(123)(124) = (13)(24)$ so all permutations of type $(2, 2)$ are in the derived subgroup. Alternative way of finishing once we have the 3-cycles: the derived subgroup is a subgroup of A_4 with at least 9 elements so it is A_4.

For the next step: $V = \{e, (12)(34), (13)(24), (14)(23)\}$ is a normal subgroup of A_4 with abelian quotient, and $(123)(124)(132)(142) =$

(12)(34), so the derived subgroup of A_4 is V. Since V is abelian the derived series is $S_4, A_4, V, 1$ and is not a composition series since V is not simple.

We can refine to a composition series by inserting (eg) $\{e, (12)(34)\}$, then the composition factors are C_2 (3 times) and C_3 (1 times).

S_4 is not nilpotent because it has non-normal Sylow subgroups (or if you prefer it is not the product of its Sylow subgroups). We could also do it by direct calculation.

(3) Prove that if G is a non-trivial finite p-group then $[G, G] < G$. Hint: Use $Z(G) \neq 1$ to power an induction.

Taking the hint, let G be a non-trivial finite p-group. If $Z(G) = G$ then G is abelian and $[G, G] = 1 < G$, otherwise $G/Z(G)$ is a non trivial p-group in the scope of the induction hypothesis, so $[G/Z(G), G/Z(G)] = [G, G]Z(G)/Z(G) < G/Z(G)$, hence $[G, G] < G$.

(4) Suppose that G is a simple group of order 60. Derive as much information about the Sylow p-subgroups of G as you can: their number, their structure, their normalisers. Hint: You can use HW1Q5 to get some information.

HW1Q5 tells us that there are no proper subgroups of index less than 5, so that for each relevant prime p the number n_p of Sylow p-subgroup must satisfy $n_p > 4$.

n_5 is a factor of 12 which is greater than 4 and congruent to 1 mod 5, so $n_5 = 6$. The Sylow 5-subgroups have type C_5 and any two have trivial intersection, so there are exactly 24 elements of order 5. The normaliser of a Sylow 5-subgroup will have order 10.

Similarly n_3 is a factor of 20 which is greater than 4 and congruent to 1 mod 3, so $n_3 = 10$. The Sylow 3-subgroups have type C_3, so there are 20 elements of order 3. The normaliser of a Sylow 5-subgroup will have order 6.

n_2 and the Sylow 2-subgroups (which have order 4) seem a bit more mysterious. $n_2 > 4$ and n_2 is an odd factor of 15, so $n_2 = 5$ (in which case normalisers have order 12) or $n_2 = 15$ (in which case each subgroup of order 4 is its own normaliser).

This is about as far as we can go just using the numerical data from Sylow's theorem. With more work we can show that $G \simeq A_5$, which tells us everything.

(5) Prove that if p and q are distinct primes there is no simple group of order p^2q.

Suppose G is a such a group. Then $n_p > 1$ and n_p divides q, so $n_p = q$ and hence q is congruent to 1 mod p, in particular $q > p$. Also $n_q > 1$ and divides p^2, and since $p < q$ it is not possible that p is congruent to 1 mod q, so $n_q = p^2$. Hence there are $p^2(q-1)$ elements of order q, leaving only p^2 elements, hence there is a unique Sylow p-subgroup and $n_p = 1$. Contradiction.

(6) Let S be the subgroup of $\Sigma_{\mathbb{N}}$ generated by the set of transpositions. Prove that $S \neq \Sigma_{\mathbb{N}}$.

S is countable but $\Sigma_{\mathbb{N}}$ is uncountable. In fact S is equal to the set of permutations which only move a finite set of elements.

Let A be the subgroup of $\Sigma_{\mathbb{N}}$ generated by the set of products of two transpositions, prove that $A = [S, S]$, $[S : A] = 2$ and A is simple. Hint: It is a standard fact that A_n is simple for $n \geq 5$, this statement and/or its proof may be helpful.

Breaking with our usual convention we let $\mathbb{N} = \{1, 2, \ldots\}$ and also identify S_n with a subgroup of $S_\mathbb{N}$ in the natural way. Using standard facts about S_n and A_n it is easy to see that $S = \bigcup S_n$, $A = \bigcup A_n$ and so $A \cap S_n = A_n$.

Since the S_n form an increasing chain, $\{[a, b] : a, b \in S\} = \bigcup_n \{[a, b] : a, b \in S_n\}$, and since the sets $\{[a, b] : a, b \in S_n\}$ form an increasing chain we have $[S, S] = \bigcup_n [S_n, S_n] = \bigcup_n A_n = A$. Routine calculation shows that $(12) \notin A$ and $S = A \cup (12)A$, so $[S : A] = 2$.

Finally let $N \lhd A$ and observe that $N \cap A_n \lhd A_n$, so that for all $n > 4$ we have $N \cap A_n = 1$ or $N \cap A_n = A_n$. Note that the groups $N \cap A_n$ form an increasing chain: so if $N \cap A_n = 1$ for infinitely many n then $N = 1$ while if $N \cap A_n = A_n$ for infinitely many n then $N = A$.

(7) Let G be the group of symmetries of the Euclidean plane. Prove that G is solvable. Hint: You may find it helpful to note that if S and T are symmetries then STS^{-1} is the map which moves $S(P)$ to $S(T(P))$ for each point P, so in some sense it's just a shifted version of T. Optional not for credit brainteaser: Is the symmetry group of Euclidean 3-space solvable?

It is a standard fact that the symmetries are rotations, translations, reflections and glide reflections: note that the first two types are orientation-preserving and the second two types are orientation reversing.

Let $G_0 = G$ and let G_1 be the subgroup of orientation preserving symmetries. A little thought shows that G_0/G_1 has type C_2. Now let G_2 be the subgroup of translations: for any

$\sigma \in G_1$ the coset σG_2 contains a unique element which fixes the origin, and so is a rotation about the origin: it follows that G_1/G_2 is isomorphic to the abelian group of rotations about the origin. Since G_2 is abelian we have a subnormal series with abelian quotients.

Brain teaser: The answer is no. One amusing proof: the symmetry group contains as a subgroup the group of rotational symmetries of an icosahedron, which is isomorphic to A_5 and so is not solvable.