ALGEBRA HOMEWORK SET 3

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Due by class time on Wednesday 28 September. Homework must be typeset and submitted by email as a PDF file.

(1) Prove that a finite group G is solvable if and only if its composition factors are cyclic groups of prime order.

(2) Compute the derived series of S_4. Hint: Keep in mind that $[G, G]$ is the smallest normal subgroup of G with an abelian quotient, this can save you computing too many commutators. Is the derived series of S_4 a composition series? Find the composition factors of S_4. Is S_4 nilpotent?

(3) Prove that if G is a non-trivial finite p-group then $[G, G] < G$. Hint: Use $Z(G) \neq 1$ to power an induction.

(4) Suppose that G is a simple group of order 60. Derive as much information about the Sylow p-subgroups of G as you can: their number, their structure, their normalisers. Hint: You can use HW1Q5 to get some information.

(5) Prove that if p and q are distinct primes there is no simple group of order p^2q.

(6) Let S be the subgroup of $\Sigma_{\mathbb{N}}$ generated by the set of transpositions. Prove that $S \neq \Sigma_{\mathbb{N}}$. Let A be the subgroup of $\Sigma_{\mathbb{N}}$ generated by the set of products of two transpositions, prove

that $A = [S, S]$, $[S : A] = 2$ and A is simple. Hint: It is a standard fact that A_n is simple for $n \geq 5$, this statement and/or its proof may be helpful.

(7) Let G be the group of symmetries of the Euclidean plane. Prove that G is solvable. Hint: You may find it helpful to note that if S and T are symmetries then STS^{-1} is the map which moves $S(P)$ to $S(T(P))$ for each point P, so in some sense it's just a shifted version of T. Optional not for credit brainteaser: Is the symmetry group of Euclidean 3-space solvable?