ALGEBRA HOMEWORK SET 2

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Due by class time on Wednesday 14 September. Homework must be typeset and submitted by email as a PDF file.

(1) Let H and N be groups and let ψ be a HM from H to $Aut(N)$. Define a binary operation on the set of pairs (h, n) with $h \in H$ and $n \in N$ by

$$(h_1, n_1)(h_2, n_2) = (h_1 h_2, \psi(h_2^{-1})(n_1) n_2)$$

(a) Prove that this operation makes the set of pairs into a group G.

This is quite routine, but you should be careful about one point. The inverse of the pair (h, n) is not (h^{-1}, n^{-1}) but $(h^{-1}, \psi(h)(n^{-1}))$.

(b) Prove that if $H' = H \times 1$ and $N' = 1 \times N$ then $H' \leq G$, $N' \lhd G$, $H' \simeq H$ and $N' \simeq N$.

This is routine.

(c) Prove that $G = H'N'$ and $H' \cap N' = 1$.

It is clear that $H' \cap N' = 1$. For the other part just observe that

$$(h, e)(e, n) = (h, n).$$

(2) Suppose that (as in a question from last week) we have $H \leq G$, $N \lhd G$, $G = HN$ and $H \cap N = 1$. We saw that every element of G can be written as hn for unique $h \in H$ and $n \in N$. Prove that

G is isomorphic to the group constructed in Q1 for a suitable choice of HM ψ.

This is a simple calculation:

$$h_1 n_1 h_2 n_2 = (h_1 h_2)(h_2^{-1} n_1 h_2 n_2),$$

where $h_2^{-1} n_1 h_2 n_2 \in N$. So setting $\psi(h)$ to be "conjugate by h" we are exactly in the situation of Q1.

(3) Let $G = S_5$. For each relevant prime p:

(a) Describe and count the elements of G whose order is a power of p.

Elements of order 2 are the 10 2-cycles and the 15 elements of cycle type $(2, 2)$ (products of disjoint 2-cycles). Elements of order 4 are the 30 4-cycles.

Elements of order 3 are the 20 3-cycles.

Elements of order 5 are the 24 5-cycles.

Sanity check: $1 + 10 + 15 + 30 + 20 + 24 = 100$, the elements we did not count are those of type $(2, 3)$ (the order 6 elements) and there are 20 of these.

(b) Describe the Sylow p-subgroups of G, and determine how many there are.

The Sylow 2-subgroups have order 8. Clearly S_5 contains a copy of the symmetry group of the square (D_4), for example

$$H = \{e, (12)(34), (14)(23), (13)(24), (13), (24), (1234), (1432)\}.$$

The other Sylow 2-subgroups are the conjugates of H.

A little thought shows that each conjugate of H is determined by the inverse pair of 4-cycles which it contains so we will have 15 2-subgroups. We will verify this in several other ways. By Sylow the number n_2 of Sylow 2-subgroups must be 1, 3, 5 or 15. The answer must be at least 15 to have a chance of getting all the elements of order 4, so it is exactly 15. Also we can use the normaliser calculation from the last part, and by this part we should expect H to be its own normaliser.

The Sylow 3-subgroups have order 3, each contains an inverse pair of 3-cycles and any two have trivial intersection, so $n_3 = 10$. We expect the normalisers to have order 12.

The Sylow 5-subgroups have order 5, each contains 4 5-cycles and any two have trivial intersection, so $n_5 = 6$. We expect the normalisers to have order 20.

(c) Describe the normalisers of the Sylow p-subgroups of G (it is enough to compute the normaliser of one Sylow p-subgroup, the others will be its conjugates)

Let H be the typical Sylow 2-subgroup as above. By the nature of conjugation in S_n, no element of the normaliser can move 5, so the normaliser is contained in the natural copy of S_4 in S_5. Since H has prime index in this subgroup and (1324) conjugates (1234) to $(3421) \notin H$, we see that the normaliser is H (as predicted).

Now let $H = \{e, (123), (132)\}$. Elements of the normaliser are clearly those which permute $\{1, 2, 3\}$, so the normaliser has structure $S_3 \times C_2$.

Finally let $H = \{e, (12345), (13524), (14253), (15432)\}$. We can view it as the normal subgroup of "rotations" in the dihedral group K generated by (12345) and (25)(43), so $K \leq N_G(H)$. One can check that nothing else is there.

(4) Let G be a group of order 20. Prove that:
   (a) G has a normal subgroup N of order 5.
   (b) G has a subgroup H of order 4.
   (c) $G = HN$ and $H \cap N = 1$.

   Use the information from the previous questions to classify groups of order 20 up to isomorphism, then check your answer against a table of groups of small order.

   This one is quite painful!

   Sylow implies that there is a normal Sylow 5-subgroup N of order 5, and a Sylow 2-subgroup of order 4. By Lagrange $H \cap N = 1$, we also know that $HN \leq G$ and $|HN| = 4 \times 5 = 20$, so $G = HN$. We are now in the situation of questions 1 and 2.

   Note that H and N are abelian, with $N \simeq C_5$ and $H \simeq C_4$ or $H \simeq C_2^2$. If $\psi = id$ then $G \simeq H \times N$ and we pick up the abelian groups $C_4 \times C_5$ and $C_2^2 \times C_5 \simeq C_2 \times C_{10}$.

   For the harder case where $\psi \neq id$ we need to understand $Aut(C_5)$. If $C_5 = \langle a \rangle$ then every AM moves a to one of the generators a, a^2, a^3, a^4, and the AM is determined by the image of a (and easily for each generator we get an AM). So $Aut(C_5)$ is isomorphic to $\{1, 2, 3, 4\}$ under multiplication mod 5, and this group is cyclic with 2 as a generator.

   Case 1: $H \simeq C_4$. Say $H = \langle h \rangle$ and $N = \langle n \rangle$. A priori there are 3 possibilities for $\psi \neq id$, but since H has an automorphism which exchanges h and h^3 there are actually only 2: in terms of conjugation they are $n^h = n^2$ and $n^h = n^4 = n^{-1}$. These will give non-isomorphic groups because the conjugation action of H on N looks different in the two cases (in one case we can induce an AM of order 4, in the other case not).

Case 2: $H \simeq C_2^2$. Say $H = \langle a, b\rangle$ and $N = \langle n\rangle$. We can use the fact that H has AM's permuting a, b and $c = ab$ arbitrarily. We know that at least one of a, b, c must induce the non-trivial AM of order 2, say $n^a = n^4 = n^{-1}$. Either b induces this AM and hence c induces id, or b induces id and c induces it. So WLOG $n^a = n^b = n^{-1}$, $n^c = n$.

Note: we know this group must be D_{10} because that group has no elements of order 4, but it's good to see it explicitly. Since n commutes with c, we see that nc has order 10, and easily $(nc)^a = n^{-1}c = (nc)^{-1}$.

This gives 5 groups, which is right.

(5) Let G be a group with distinct elements a and b such that $|a| = 2$, $|b| = n$ and $b^a = b^{-1}$. Prove that $a \notin \langle b\rangle$, and that the subgroup generated by a and b is dihedral of order $2n$.

Routine. b corresponds to a rotation through $2\pi/n$ and a to a reflection.

(6) Let G be a nonabelian group of order 8. Prove that G has at least one element of order 4 and no element of order 8.

If all elements of G have order one or two, $a^2 = e$ for all a and so G is abelian. If there is an element of order 8 then G is cyclic. Hence there is an element of order 4.

We saw in HW1 that $Z(G) \simeq C_2$ and $G/Z(G) \simeq C_2 \times C_2$. Let $Z(G) = \langle a\rangle$. Prove that for any element b of order 4, $b^2 = a$ and $\langle b\rangle \lhd G$.

$(bZ(G))^2 = Z(G)$ because $G/Z(G) \simeq C_2^2$, so $b^2 \in Z(G)$. Since b^2 has order 2 we see $b^2 = a$. Since $\langle b \rangle$ has index 2 it is normal.

Now fix an element b of order 4, and an element $c \notin \langle b \rangle$. Prove that $G = \langle b, c \rangle$, $b^c = b^{-1}$. Prove that if c has order 2 then G is dihedral of order 8.

Thinking about cosets makes it clear that $G = \langle b, c \rangle$. b^c must lie in $\langle b \rangle$ by normality and must have order 4. If $b^c = b$ then b commutes with c and G is abelian, so $b^c = b^{-1}$ as claimed.

Now suppose that c has order 4. Prove that G is not dihedral, a is the only element of order 2, bc has order 4, $cb = abc$, and $G = \{e, b, c, bc, a, ab, ac, abc\}$.

D_4 has only 2 elements of order 4 so G is not dihedral. Any element outside $\langle b \rangle$ could play the role of c, and if it had order two then G would be dihedral, so a is the only element of order 2. $bc \neq e$ because $c \notin \langle b \rangle$, $bc = a = b^2$ implies $c = b$, so bc has order 4. Since $b^c = b^{-1} = b^3$, $cb = b^3c = b^2bc = abc$. Finally $\langle b \rangle = \{e, a, b, b^3 = ab\}$, and $G = \langle b \rangle \cup \langle b \rangle c$.

Consult a table of groups of small order to see if there is a non-dihedral non-abelian group of order 8 (you have proved enough to show that there is at most one such group).

There is such a group, the "quaternion group". To see the isomorphism take (for example) $a = -1$, $b = i$, $c = j$, $bc = k$.