(1) Before proof of JH, a few trivial remarks.
   (a) If \( \{ G_i \} \) is a subnormal series \( \{ G_i \cap H \} \)
       in \( H \leq G \), and a subnormal series \( \{ G_i/N \} \) in \( G/N \) for \( N \triangleleft G \).
   (b) If \( G \) is simple and \( \{ G_i \} \) is a subnormal series then there is \( k \) such that
       \( G_j = \{ e \} \) for \( j > k \), \( G_j = G \) for \( j \geq k \).

(2) Proof of JH: show by induction on \( m \) that if \( \{ H_i : 0 \leq i \leq m \} \) a composition
     series for \( G \) then any other series has same length and “same” quotients.

   \( m = 1 \). \( G \) is simple which makes it easy.

   Let \( m \geq 1 \) and assume we have established the IH for \( m \). Consider a CS
   \( \{ H_i : 0 \leq i \leq m + 1 \} \) for \( G \) and let \( H = H_m \), so that \( \{ H_i : 0 \leq i \leq m \} \) is a
   CS for \( H \). Note that \( e < H < G \). Now let \( \{ G_j : 0 \leq j \leq n + 1 \} \) be another CS,
   where \( n \geq 1 \) since \( G \) not simple. Choose \( k \) least such that \( G_k \not\triangleleft H \) and
   note that \( 0 < k \leq n + 1 \).

   We will show that \( G_k/G_{k-1} \cong G/H \) and produce a CS for \( H \) whose
   quotients are IMic to \( G_j/G_{j-1} \) for \( j \neq k \). The IH then shows that \( m = n \)
   and this new CS for \( H \) has quotients IMic to \( H_i/H_{i-1} \) for \( i \neq m + 1 \), so we
   are done.

   Note that by the minimal choice of \( k \), \( G_j \cap H = G_j \) for \( j < k \). Also
   \( \{ G_j \cap H \} \) is a subnormal series in \( H \). Our desired new composition series
   for \( H \) will be made by taking \( \{ G_j \cap H \} \) and deleting the entry \( G_k \cap H \). In
   fact we will show that
   (a) \( G_j/G_{j-1} \cong (G_j \cap H)/(G_{j-1} \cap H) \) for \( j > k \).
   (b) \( G_k \cap H = G_{k-1} \cap H = G_{k-1} \).
   (c) \( G/H \cong G_k/G_{k-1} \).

   The sequence \( \{ G_j/H \} \) is subnormal in the simple group \( G/H \), so by
   choice of \( k \) we have \( G_j/H = G \) for \( j \geq k \). So for \( j \geq k \) \( G_j/H = G_{j-1}/H \cong
   G_{j-1}/(G_j \cap H) \), and the quotient \( G_j/(G_j \cap H) \) is simple.

   Now let \( j \geq k \) and observe that \( G_{j-1} \) and \( G_j \cap H \) are normal in \( G_j \), so
   that \( G_{j-1} \cap (G_j \cap H) < G_j \) and thus \( G_{j-1}/(G_j \cap H) < G_j/(G_j \cap H) \).
   It follows by simplicity that \( G_{j-1}/(G_j \cap H) \) is either \( G_j \cap H \) or \( G_j \). In
   the case that \( j > k \) we know that \( G_j \not\leq H \) and the first case fails, so
   that \( G_j = G_{j-1}(G_j \cap H) \); it follows that for \( j > k \) we have \( G_j/G_{j-1} =
   G_{j-1}/(G_j \cap H) \), so \( G_j \) is subnormal in \( G_j \cap H \).

   Now we note that \( G_{k-1} \) and \( G_k \cap H \) are both subgroups of \( H \) and
   normal in \( G_{k-1} \), so that \( G_{k-1}/G_{k-2} \leq H \) and \( G_{k-1}/G_{k-2} < G_k \). So
   \( G_k/G_{k-1} \triangleleft H \). But in the latter case we have \( G_k \not\leq H \) in contradiction to choice
   of \( k \), so that \( G_k \cap H = G_{k-1} \) and \( G_k \cap H \triangleleft G_{k-1} \). Since \( G_{k-1} \) is a
   subgroup of both \( G_k \) and \( H \), we see \( G_k \cap H = G_{k-1} \cap H = G_{k-1} \).
Finally we recall that $G = G_jH$ for $j \geq k$, in particular $G = G_kH$. So $G/H = G_kH/H \cong G_k/(G_k \cap H) = G_k/G_{k-1}$.

3) If $G$ is finite, solvable, simple and non-trivial then it’s cyclic of prime order.

4) If $G$ has a composition series then the composition factors are the (IM classes) of simple groups which appear as quotients.

5) Let $G$ be finite. $G$ solvable iff its factors are cyclic of prime order.

6) Sketch of a proof that $A_n$ is simple for $n \geq 5$.

First show that $A_n$ is generated by 3-cycles, and that all 3-cycles are conjugate in $A_n$. Then take $N < A_n$ non-trivial and argue it contains a 3-cycle; let $\sigma \in N$ be a non-identity element where $\{i : \sigma(i) \neq i\}$ has minimal size, and argue that $\sigma$ must be a 3-cycle by showing that o/w we can contradict minimality.