(1) Last time we analysed $\Gamma(F/E)$ where $E = \mathbb{Z}/p\mathbb{Z}$ and $F$ is an alg closure. Now we ask whether there is a notion of Galois correspondence? The answer is yes but with a wrinkle.

(2) Recall that $F = \bigcup F_n$ where $F_n/F$ is a finite Galois extension. Say that $X \subseteq \Gamma(F/E)$ is open iff for all $\sigma \in X$ there is $n$ such that $\forall \tau \in \Gamma(F/E)$ $\tau \upharpoonright F_n = \sigma \upharpoonright F_n \implies \tau \in X$. $X$ is closed iff its complement is open, that is to say that if $\sigma \in \Gamma(F/E)$ is such that for all $n$ there is $\tau \in X$ with $\sigma \upharpoonright F_n = \tau \upharpoonright F_n$, then $\sigma \in X$.

Cultural notes: This defines a topology on $\Gamma(F/E)$. The topology makes it into a compact topological group, with a basis of nbhds of $e$ consisting of open subgroups.

(3) Note that if $E'$ is an intermediate field then easily $E' = \bigcup \{F_n : F_n \subseteq E'\}$. It follow easily that if $H = \Gamma(F/E')$ then $H$ is a closed subgroup of $\Gamma(F/E)$. It is not too hard to check that not all subgroups are closed, and that the maps $H \mapsto \text{Fix}(H)$ and $E' \mapsto \Gamma(F/E')$ set up a bijection between closed subgroups and intermediate fields; moreover normal subgroups correspond to normal extensions.

(4) The same kind of analysis works for the alg closure $\bar{\mathbb{Q}}/\mathbb{Q}$, but the group is much more complicated. To be a bit more explicit: let $I$ be the set of $F \subseteq \bar{\mathbb{Q}}$ such that $F/\mathbb{Q}$ is finite Galois. Then every $\sigma \in \Gamma(\bar{\mathbb{Q}}/\mathbb{Q})$ is determined by $\{\sigma \upharpoonright F : F \in I\}$ and conversely any compatible family $\{\tau_F \in \Gamma(F/\mathbb{Q}) : F \in I\}$ gives a unique $\tau$ with $\tau \upharpoonright F = \tau_F$. As above we can introduce the notion of closed subgroup using now $I$ in place of $\{F_n\}$, and get a Galois correspondence.

(5) (Did not say this in class) The whole story works in the following general setting: $E$ is a field, $E^{sep}/E$ is a separable closure of $E$, and $I$ is the set of $F \subseteq E^{sep}$ such that $F/E$ is finite Galois.

(6) Now to get a bit more down to earth. We analyse the *cycloomatic extensions* of $\mathbb{Q}$, that is extensions obtained by adding roots of $1$.

Let $\alpha = e^{2\pi i/n}$. Then as we saw $\mathbb{Q}(\alpha)$ contains the primitive $n$th roots of $1$ which are the elements of form $\alpha^j$ where $gcd(j, n) = 1$ and $0 < j < n$. By definition there are $\phi(n)$ such $j$, where $\phi$ is Euler’s totient function.

We will show that $\alpha$ has degree $\phi(n)$ over $\mathbb{Q}$, and also the roots of its minimal polynomial are exactly the primitive $n$th roots of $1$.

(7) Let $n = p$ prime. Then $\alpha$ is a root of $f = \sum_{j < p} x^j = (x^p - 1)/(x - 1)$. We claim that $f$ is irreducible. To see this apply to $f$ the unique AM of the ring $\mathbb{Z}[x]$ which fixes $\mathbb{Z}$ and moves $x$ to $x + 1$, then $f$ is mapped to

\[
\left(\frac{x^p + 1}{x} - 1\right)/x = x^p + \binom{p}{1} x^{p-1} + \ldots + \binom{p}{p-1}.
\]
which is easily irreducible by Eisenstein.