A Kummer extension of $E$ is a sf for a poly $\prod_i (x^{n_i} - a_i)$. Easy exercise: if $E$ has char zero, any K extn is Galois and has solvable Galois group. (Hint: induction on the number of factors, proof of main result later in today’s note is similar)

(2) For simplicity, the whole discussion of solvability of polynomial equations is done in characteristic zero. Adapting it to char $p$ is not too hard.

(3) An extension by radicals of $E$ is $F/E$ such that there exist $E_0 \subseteq E_1 \subseteq \ldots \subseteq E_k = F$ and $\alpha_i \in E_{i+1}$, $n_i > 0$, $a_i \in E_i$ st $\alpha_i^{n_i} = a_i \in E_i$ and $E_{i+1} = E_i(\alpha_i)$ for $i < k$.

(4) A polynomial $f \in E[x]$ is solvable by radicals iff there are $F/E$ and $G/F$ so that $F/E$ is a spl field for $f$ and $G/E$ is an extn by radicals.

(5) Goal for today: Thm: Let char of $E$ be zero. If $f$ is sol by radicals and $F$ is the spl field then $\Gamma(F/E)$ is solvable.

(6) We will show a technical lemma: Let char of $E$ be zero. For any $G/E$ an extn by radicals there is $G/G$ such that $G/E$ is Galois with a solvable Galois group.

This suffices for the proof of Thm. For let $F/E$ be spl field for $f$ and $G/F$ be st $G/E$ is extn by radicals, appeal to Lemma to get $G$ as above. By the Fund Thm, since $F/E$ is Galois, $\Gamma(F/E)$ is IMic to a quotient of the solvable group $\Gamma(G/E)$, hence is solvable.

(7) To prove lemma: we know there exist $E_0 \subseteq \ldots \subseteq E_k = G$ and $\alpha_i \in E_{i+1}$, $n_i > 0$, $a_i \in E_i$ st $\alpha_i^{n_i} = a_i \in E_i$ and $E_{i+1} = E_i(\alpha_i)$ for $i < k$.

If $k = 0$ OK. If $k > 0$ then by induction we may find $E_{k-1}/E$ which is Galois with solvable Galois group. Fix some $g \in E[x]$ so that $E_{k-1}/E$ is a spl field extn for $g$.

Now define $h = \prod_{\sigma \in \Gamma(E_{k-1}/E)} (x^{nk_i} - \sigma(a_{k-1}))$. Any $\tau \in \Gamma(E_{k-1}/E)$ permutes the factors in this product, so $\tau(h) = h$. But $E_{k-1}/E$ is Galois so $h \in E[x]$. Define $E_k/E_{k-1}$ to be a sf for $h$, and note that easily $E_k/E_{k-1}$ is a sf for $gh$, in particular it is a Galois extension.

Now let $m = m_{E_{k-1}}$. This divides $x^{nk_i} - a_{k-1}$, so it divides $h$. Choosing any $\beta \in E_k$ such that $m(\beta) = 0$, there is an IM from $E_k = E_{k-1}(\alpha_k)$ to $E_{k-1}(\beta)$ fixing elts of $E_{k-1}$. So we may assume WLOG that $E_k \subseteq E_{k-1}$.

Finally $\Gamma(E_{k-1}/E)$ is solvable by the indn hypothesis. $\Gamma(E_k/E_{k-1})$ is the group of a spl field extension for $h$, which is a Kummer extn, so this is also a solvable group. Since $E_{k-1}/E$ is a Galois extn, by the Fund Thm $\Gamma(E_{k-1}/E) \cong \Gamma(E_k/E)/\Gamma(E_k/E_{k-1})$. So by easy group theory since the normal subgroup $\Gamma(E_k/E_{k-1})$ and the quotient group $\Gamma(E_{k-1}/E)$ are both solvable, $\Gamma(E_k/E)$ is solvable.