(1) We will show that every simple L artinian ring is of the form $\text{End}_D(M)$ for $D$ a division ring and $M$ a (finite dimensional) left $D$-module.

(2) When $M$ is a left $R$-module we may set $S = \text{End}_R(M)$, and make $M$ into a left $S$-module by defining $\beta m = \beta(m)$. For each $r \in M$ we let $\lambda_r : m \mapsto rm$, then in general $\lambda_r$ may not be in $\text{End}_R(M)$ but is in $\text{End}_S(M)$.

Moreover if we define $\phi : r \mapsto \lambda_r$ then $\phi$ is a ring HM from $R$ to $\text{End}_S(M)$.

(3) (Rieffel’s lemma) Let $R$ be simple, let $M$ be a nonzero left ideal of $R$. Define $S$, $\lambda_R$, $\phi$ as above. Then $\phi$ is a ring HM between $R$ and $\text{End}_S(M)$.

Proof: $\phi(1) = id_M$ which is not the zero map, so ker($\phi$) $\neq R$, so by simplicity ker($\phi$) $= \{0\}$ and $\phi$ is injective.

We claim that $\phi[M]$ is a left ideal of $\text{End}_S(M)$. To see this let $m \in M$ and $\alpha \in \text{End}_S(M)$, we will show that in fact $\alpha \lambda_m = \lambda_{\alpha(m)}$. let $u \in M$, then by definition $(\alpha \lambda_m)(u) = \alpha (mu)$. Since $u \in M$, the map $\beta : x \in M \rightarrow xu$ is a map from $M$ to $M$, and we may verify that $\beta \in \text{End}_R(M) = S$. So by $S$-linearity of $\alpha$, $\alpha (mu) = \alpha (\beta m) = \beta (am) = \alpha (m)u$, that is $\alpha \lambda_m = \lambda_{\alpha(m)}$.

Now $MR$ is a nonzero two sided ideal of $R$, so $R = MR$ and thus $\phi[R] = \phi[MR]$. It follows easily that $\phi[R]$ is a left ideal of $\text{End}_S(M)$, and since $\lambda_1 = id_M \in \phi[R]$, we see that $\phi[R] = \text{End}_S(M)$.

(4) Let $R$ be a nonzero simple L artinian ring. Then $R \simeq \text{End}_D(M)$ for some division ring $D$ and some FD left $D$-module $M$.

Proof: we may as well assume that $R$ is nonzero. By the L artin property we may find a minimal L ideal $M$. $M$ is a nonzero simple left $R$-module so if we set $D = \text{End}_R(M)$ then $D$ is a division ring. By Rieffel $R$ is isomorphic to $\text{End}_D(M)$.

To finish we need to see that $M$ is FD as a left $D$-module. But it is easy to see (next HW) that if $M$ is an infinite dimensional left $D$-module then $\text{End}_D(M)$ is not Artinian.

(5) Remark: a very mild generalisation of last week’s homework shows that if $M$ is a FD left $D$-module and $D$ is a division ring then in fact $\text{End}_D(M)$ is a left (and right) artinian (and noetherian) simple ring.