(1) Goals for this part of the course: A) some structure theory of non commutative rings (analysis of the simple and semisimple rings). B) elementary facts about commutative rings and their modules (Basissatz, fg modules over a PID). If time allows I might talk about integrality and ring of integers in a “number field”.

(2) Convention: until further notice all rings are unital and so are all modules (a left $R$-module is unital if $1m = m$ for all $m$, similarly $m1 = m$ for right $R$-modules.

(3) I develop the theory of left $R$-modules, and leave it to you to adjust for right and two-sided modules.

(4) If $N$ is a left $R$-module then $M$ is a submodule iff $M \leq (N, +)$ and $RM \subseteq M$, where $RM = \{ rm : r \in R, m \in M \}$. $M$ inherits a module structure. We write $M \leq N$.

(5) If $X \subseteq M$ then a linear combination of elements of $X$ is an element of form $\sum_{i=1}^{m} r_{i}x_{i}$ for $r_{i} \in R, x_{i} \in X$. The submodules are precisely the sets closed under linear combinations. $\langle X \rangle$ is the set of all LCs of elements of $X$, this is the least submodule containing $X$. Note: this is just generalised linear algebra, with the field $F$ replaced by a general ring $R$.

(6) Module $M$ is fg iff it is of form $\langle X \rangle$ for some finite $X \subseteq M$. Analogy with Lin Alg is misleading here: submodules of an fg module need not be fg, and also not every module has an independent generating set. See the discussion of Noetherian and free modules later for more on these issues.

USEFUL IDEA: can always look at a minimal generating set!

(7) If $M \leq N$ we can form a quotient module $N/M$. Elements are left cosets $n + M$ (same as the quotient of abelian groups). Scalar mult given by $r(n + M) = rn + M$. Again it generalises quotient spaces from Lin Alg.

(8) HM of left $R$-modules $\phi : M \rightarrow N$ is a HM of groups from $(M, +)$ to $(N, +)$ such that $\phi(rm) = r\phi(m)$ all $r, m$. This is equivalent to linearity in the obvious sense.

(9) First IM theorem. Let $\phi : M \rightarrow N$ be HM, as usual $\ker(\phi) = \{ m : \phi(m) = 0 \} \leq M$ and $\mathrm{im}(\phi) = \{ \phi(m) : m \in M \} \leq N$. Then $M/\ker(\phi) \simeq \im(\phi)$ by an IM in which $m + \ker(\phi) \mapsto \phi(m)$. Proof: we have this for the modules considered as groups under +, just check that the IM is an IM of modules too.

(10) If $M, N \leq P$ then $M + N, M \cap N \leq P$ and $(M + N)/N \simeq M/(M \cap N)$.

(11) External direct sum: $M \oplus N$ is $M \times N$ with coordinatewise operations.

(12) Internal direct sum: Let $M, N \leq P$ with $M + N = P$, $M \cap N = \{0\}$. Then $(m, n) \mapsto m + n$ is an IM from $M \oplus N$ to $P$.

(13) External direct sum again: let $M_{i}$ for $i \in I$. Then $\bigoplus_{i \in I} M_{i}$ is the subset of $\prod_{i \in I} M_{i}$ consisting of $f$ such that $\{ j : f(j) \neq 0_{M_{j}} \}$ is finite. This is easily
seen to be a module under coordinatewise operations. Exercise: what is the internal version?

(14) $R$ is a left $R$-module (with scalar multiplication $\times_R$), and submodules are precisely the left ideals. If $I$ is a left ideal then $R/I$ makes sense as a quotient of $R$-modules, so it has an $R$-module structure.

(15) Module $M$ is cyclic iff $M = Rm$ for some $m$. The map $r \mapsto rm$ is a module homomorphism so $Rm \simeq R/\text{Ann}(M)$, where $\text{Ann}(m) = \{r : rm = 0\}$ is a left ideal called the annihilator of $m$. 