MIDTERM

Due by midnight on Sat March 10. You may not collaborate but may consult any
printed or online source (please credit these sources). Please email your completed
final to me (not Yimu).

Attempt exactly seven questions.

(1) Let $G$ be a finite nilpotent group and let $N \neq \{e\}$ with $N \lhd G$. Show that
$N \cap Z(G) \neq \{e\}$.

Recall that since $G$ is nilpotent it has a normal series $G_i$ such that
$[G, G_{i+1}] \leq G_i$. Let $i$ be least such that $N \cap G_{i+1} \neq \{e\}$. Now $[G, N] \leq N$
since $N$ is normal, so $[G, N \cap G_{i+1}] \subseteq N \cap G_i = \{e\}$. Hence $N \cap G_{i+1} \subseteq Z(G)$
and we are done.

(2) Let $k$ be a field and let $V$ be a VS over $k$ with $\dim_k(V)$ infinite. Show that
the ring $\text{End}_k(V)$ is not left Artinian. Is it simple?

For any subspace $W$ of $V$, consider the set of $\phi \in \text{End}_k(V)$ such that
$\phi \upharpoonright W = 0$. It is easy to see that this is a left ideal. Let $B$ be a basis
and enumerate a countably infinite subset of $B$ as $b_0, b_1, b_2, \ldots$. For each
$n$ let $W_n$ be the subspace spanned by $\{b_i : i < n\}$ and let $I_n$ be the ideal
$\{\phi : \phi \upharpoonright W_n = 0\}$. Clearly $I_n \subseteq I_{n+1}$, it remains to see that the inclusion is
strict. Let $\phi$ be the unique linear map which takes $b_i$ to zero for $i < n$ but
fixes all other elements of the basis. Then $\phi \in I_n$ but $\phi \notin I_{n+1}$.

For simplicity we observe that the set of endomorphisms $\phi$ which have
a finite dimensional range is a proper nonzero two-sided ideal, so that the
ring is not simple. How did I discover this funny example? We can think
of the endomorphism ring as a ring of matrices which have finitely many
entries in each column (think about it), and then the ideal generated by a
matrix with a single 1 in it consists of matrices with finitely many nonzero
rows.

(3) Let $p$ be prime and let $G$ be a finite abelian group of order $p^n$ where $n > 0$.
Show that if $\Phi(G)$ is defined (as usual) to be the intersection of the maximal
subgroups of $G$ then $\Phi(G) = \{g^p : g \in G\}$.

If $G$ is any finite abelian $p$-group and $M < G$ then $G/M$ has a subgroup
of order $p$, so that if $M$ is maximal necessarily $[G : M] = p$. Then $G/M$ is
a group of order $p$, so that $(gM)^p = M$ and $g^p \in M$ for all $g$. This shows
that $\{g^p : g \in G\} \subseteq \Phi(G)$.

For the converse suppose that $g$ is not of form $h^p$. Using the structure theorem for finite abelian groups it is easy to construct a subgroup of index
$p$ which does not contain $g$.

(4) Let $R$ be an infinite ID with finitely many units. Show that there are
infinitely many maximal ideals. Would this result still be true if we dropped
the requirement on the units?
Note that $R$ is not a field so that the zero ideal is not maximal. Since $R$ is an infinite ID, all nonzero ideals (in particular all maximal ideals) are infinite. Suppose that $M_1, \ldots, M_h$ are the maximal ideals and note that $J(R) = \cap_i M_i$. Then choose $r_i \in M_i$ for $i > 1$, and let $m = \prod_i r_i$. For every $r \in M_1$, we see that $rm \in J(R)$, so that $1 + rm$ is a unit. But as $R$ is an ID the map $r \in M_1 \mapsto 1 + rm$ is 1-1 and we get infinitely many units, contradiction.

If $R$ is an infinite field then $R$ has only one maximal ideal.

(5) Let $R$ be a ring (where as usual $R$ may be non commutative, but does have a 1). Let $e$ and $f$ be idempotents. Prove that

(a) $Re = Rf$ if $f = e + (1 - e)ae$ for some $a$.

Let $Re = Rf$, so that $e = xf$ and $f = ye$. Then $ef = e$ and $fe = f$,

also $efe = e$, so $e + (1 - e)fe = e + (1 - e)f = e + f - ef = f$.

Conversely if $f = e + (1 - e)ae$ then $fe = f$, $ef = e$, so that $e + (1 - e)fe = f$, $f + (1 - f)e = e$, so $Re = Rf$.

(b) $Re$ and $Rf$ are isomorphic as left $R$-modules iff there exist $a$ and $b$

such that $e = ab$ and $f = ba$.

Suppose that $\phi$ is an IM from $Re$ to $Rf$. Note that $\phi$ is 1-1, as is $\phi^{-1}$.

Let $\phi(e) = rf$ and $\phi(se) = f$. Now $\phi(rfse) = (rf)\phi(se) = rf^2 = rf = \phi(e)$, so $e = (se)(rf)$.

Converse is similar.

(c) $eR$ and $fR$ are isomorphic as right $R$-modules if $Re$ and $Rf$ are iso-

morphic as left $R$-modules.

Use symmetry and the first two parts,

(6) Let the finite group $G$ act transitively on $X$.

Let $x \in X$, let $p$ be a prime dividing the order of $Stab(x)$ and let $P$ be a

Sylow $p$-subgroup of $Stab(x)$. Let $Fix(P)$ be the set of points fixed by $P$,

that is to say $Fix(P) = \{x : \forall g \in P \ gx = x\}$.

Prove that

(a) If $g \in N_G(P)$ and $y \in Fix(P)$ then $gy \in Fix(P)$ (to put it another

way, the action of $G$ on $X$ induces an action of $N_G(P)$ on $Fix(P)$).

(b) The action of $N_G(P)$ on $Fix(P)$ is transitive.

Let $H = Stab(x)$, so that $p$ divides $|H|$ and $P$ is a Sylow $p$-subgroup of

$H$.

Let $g \in N_G(P)$ and $y \in Fix(P)$. For all $h \in N_G(P)$ we have that

$h^{-1} \in N_G(P)$, so that $h^{-1}ph \in P$. So $h^{-1}phy = y$ as $y \in Fix(P)$, and thus

$p(hy) = hy$. We showed that $hy \in Fix(P)$ as required.

Note that since $P \leq H = Stab(x)$, $x \in Fix(P)$ and it suffices to show

that the orbit of $x$ under the action of $N_G(P)$ is $Fix(P)$. Let $y \in Fix(P)$,

then since the action of $G$ on $X$ is transitive we may choose $g \in G$ so that

$gy = x$.

For all $p \in P$, $py = y$, so easily for all $p \in P$ we have that $gpg^{-1}x = x$.

It follows that $P^g \leq Stab(x) = H$. So $P^g$ is a Sylow $p$-subgroup of $H$,

and must be conjugate in $H$ to the subggroup $P$. Fix $h \in H$ such that

$P^h = P^b$, then $P^{h^{-1}g} = P$ and $h^{-1}g \in N_G(P)$. Also $h^{-1}gy = x$, and we

are done.

(7) Let $G$ be a finite group. Prove that if $H$ is a $p$-group for some prime $p$,

Hint: Let $X$ be the set of left cosets of $H$, and consider the action of $H$ on $X$ by $h(xH) = hxH$.

Taking the hint, we will determine which of the cosets are fixed points (that is the orbit has size one). Easily $xH$ is a fixed point iff $hxH = xH$ for all $h$ iff $x^{-1}hxH = H$ for all $h$ iff $x^{-1}hx \in H$ for all $h$ iff $x^{-1} \in N_G(H)$ iff $x \in N_G(H)$. So the fixed points are the cosets $xH$ for $x \in N_G(H)$, and since $H \leq N_G(H)$ there are exactly $|N_G(H) : H|$ such cosets.

If $H = \{e\}$ then $N_G(H) = G$ so we may as well assume that $|H|$ is a positive power of $p$. Then all the non-fixed points live in orbits whose size is a power of $p$, so routinely the total number of points in $X$ (that is $|G : H|$) is congruent to the number of fixed points (that is $|N_G(H) : H|$).

(8) Let $A$ be an $n \times n$ matrix with integer entries. Consider each row as an element of the free abelian group $H = \mathbb{Z}^n$, and let $G$ be the subgroup generated by the rows. Show that

(a) $G$ has rank $n$ iff $A$ is non-singular.
(b) If $G$ has rank $n$ then $H/G$ is finite with $|H/G| = |\text{det}(A)|$.

By clearing fractions, a set of elements in $\mathbb{Z}^n$ is linearly independent over $\mathbb{Z}$ iff it is linearly independent over $\mathbb{Q}$.

We may choose a basis $x_1, \ldots, x_n$ for $H$ and integers $0 < c_1 < \ldots < c_n$ so that $c_1x_1, \ldots, c_nx_n$ is a basis for $H$. Now it is easy to find invertible integer matrices $S$ and $T$ so that $SAT$ has determinant $\prod c_i$.

(9) Let $G$ be a group (not necessarily finite) with subgroups $H$ and $N$ such that $[G : H]$ is finite, $N \trianglelefteq G$ is finite, and the numbers $[G : H]$ and $|N|$ are coprime. Prove that $N \subseteq H$.

Start by observing that $HN \leq G$ and so $H \leq HN \leq G$, in particular the index $[HN : H]$ is finite and divides $[G : H]$, so is coprime with $|N|$. Now we try and count right cosets of $H$ in $HN$.

Such cosets have the form $Hhn = Hn$ for some $h \in H$ and $n \in N$. What is more $Hn_1 = Hn_2$ iff $n_1n_2^{-1} \in H$ iff $n_1n_2^{-1} \in H \cap N$ iff $(H \cap N)n_1 = (H \cap N)n_2$. We have shown that $[HN : H] = [N : N \cap H]$, so this number is both a divisor of $|N||N$ and $[G : H]$. So $[HN : H] = 1$, $HN = H$, and $N \leq H$ as required.

(10) Find an example of a module which is not cyclic, but all fg submodules are cyclic.

$\mathbb{Q}$ as a $\mathbb{Z}$-module works. For if $a_1/b_1, \ldots, a_n/b_n$ are nonzero rationals we may as well assume that all the $b_i = b$ for fixed nonzero $b$. Then by elementary number theory the subgroup generated by the $a_i/b$ is cyclic with generator $d/b$, where $d = \gcd(a_1, \ldots, a_n)$.

(11) Find the order of the group generated by $a, b$ subject to the relations $a^6 = e$, $b^2 = a^3 = (ab)^2$.

Start by noting that $b^4 = a^6 = e$. Also as $b^2 = abab$, $ba = a^5(abab)b^3 = a^5b$. Routinely we can use this relation to write all elements as $a^ib^j$, and then use $b^2 = a^3$ and $a^6 = e$ to get that WLOG $0 \leq i < 6$ and $0 \leq j < 2$.

So the order of the group is at most 12. To finish we must exhibit a group of order at least 12 generated by elements satisfying the relations.

It is not hard to see that the subgroup of $S_{12}$ generated by

$$A = (123456)(789101112)$$
and

\[ B = (1\ 7\ 4\ 10)(2\ 12\ 5\ 9)(3\ 11\ 6\ 8) \]

has 12 elements, and that the generators satisfy the desired relations.

In case you are skeptical:

\[ A^0 = () \]

\[ A^1 = (1\ 2\ 3\ 4\ 5\ 6)(7\ 8\ 9\ 10\ 11\ 12) \]

\[ A^2 = (1\ 3\ 5)(2\ 4\ 6)(7\ 9\ 11)(8\ 10\ 12) \]

\[ A^3 = (1\ 4)(2\ 5)(3\ 6)(7\ 10)(8\ 11)(9\ 12) \]

\[ A^4 = (1\ 5\ 3)(2\ 6\ 4)(7\ 11\ 9)(8\ 12\ 10) \]

\[ A^5 = (1\ 6\ 5\ 4\ 3\ 2)(7\ 12\ 11\ 10\ 9\ 8) \]

\[ B = (1\ 7\ 4\ 10)(2\ 12\ 5\ 9)(3\ 11\ 6\ 8) \]

\[ AB = (1\ 8\ 4\ 11)(2\ 7\ 5\ 10)(3\ 12\ 6\ 9) \]

\[ A^2B = (1\ 9\ 4\ 12)(2\ 8\ 5\ 11)(3\ 7\ 6\ 10) \]

\[ A^3B = (1\ 10\ 4\ 7)(2\ 9\ 5\ 12)(3\ 8\ 6\ 11) \]

\[ A^4B = (1\ 11\ 4\ 8)(2\ 10\ 5\ 7)(3\ 9\ 6\ 12) \]

\[ A^5B = (1\ 12\ 4\ 9)(2\ 11\ 5\ 8)(3\ 10\ 6\ 7) \]

And as for the relations trivially \( A^6 = e \), and

\[ A^3 = B^2 = (AB)^2 = (1\ 4)(2\ 5)(3\ 6)(7\ 10)(8\ 11)(9\ 12) \]

It is instructive to see how all our elementary group theory plays out in this setting. Start by analysing groups of order 12. Abelian case is easy. So let \( G \) be a nonabelian group of order 12. 12 = \( 2^2 \cdot 3 \), let \( n_p \) be the number of Sylow \( p \)-subgroups. \( n_3 \) is 1 or 4 and \( n_2 \) is 1 or 3. They can’t both be 1 as then \( G \) is the product of its Sylow subgroups, hence abelian.

Suppose that \( n_3 = 4 \). So there are 4 subgroups of order 3 and each is its own normaliser. Let \( G \) act on this set of subgroups by conjugation, and note that the kernel of the action is the intersection of the normalisers, and is trivial. So \( G \) is isomorphic to a subgroup of \( S_4 \), and so easily \( G \) is \( A_4 \).

The other possibility is that \( n_3 = 1 \) and \( n_2 = 4 \). Let \( N = \langle y \rangle \) be the unique 3-subgroup, and let \( H \) be some Sylow 4-subgroup. Then \( H \cap N = \{e\}, G = HN \) and we are in the usual “semidirect product” setting. Note that \( Aut(N) = \{e, \rho\} \) where \( \rho \) is the inversion map. Since \( G \) is nonabelian the natural HM from \( H \) to \( Aut(N) \) via conjugation must be surjective, so its kernel has index 2 in \( H \).

If \( H \) is a Klein 4-group \( H = \{e, y, z, yz\} \) then we may as well assume that \( yxy^{-1} = x, zxz^{-1} = x^2 \). Then we are looking at the group \( G_2 \) with presentation \( \langle x, y, z | x^3 = y^2 = z^2 = e, yz = zy, yx = xy, zx = x^2z \rangle \). With some effort we can see that this is the dicyclic group \( D_6 \).

Let \( H = \langle y \rangle \) be cyclic of order 4, then since we are in the nonabelian case we must have \( yxy^{-1} = x^2 \). So we are looking at the group with presentation \( \langle x, y | x^3 = y^4 = e, yx = x^2y \rangle \). This is the group we are studying here, a so-called “dicyclic group”.
How did I get these weird permutations? I just assumed that the group had 12 elements and then considered how it should act on itself by left multiplication, using the labels

\[ e = 1, a = 2, a^2 = 3, a^3 = 4, a^4 = 5, a^5 = 6, b = 7, ab = 8, a^2b = 9, a^3b = 10, a^4b = 11, a^5b = 12 \]