Recall that a division ring is a ring (with 1 as usual) such that $1 \neq 0$ and every nonzero element is a unit. A field is of course a commutative division ring.

A ring $R$ is simple iff it is simple as two-sided $R$-module, that is the only two-sided ideals of $R$ are $\{0\}$ and $R$.

(1) Let $N$ be a left $R$-module. Recall that a submodule $M \leq N$ is maximal iff $M < N$ and there are no intermediate modules, equivalently $N/M$ is a non-trivial simple left $R$-module.

(a) If $I \neq R$ is a left ideal of $R$, then $R/I$ (considered as a left $R$-module) has a maximal submodule.

(b) If $M$ is a nonzero cyclic left $R$-module, then $M$ has a maximal submodule.

(c) If $m_1, \ldots, m_n$ is a generating set of minimal size for the nonzero fg left $R$-module $M$, and $M' = Rm_2 + \ldots + Rm_n$, then $M' < M$ and $M/M'$ is cyclic.

(d) Every fg nonzero left $R$-module has a maximal submodule.

(2) Let $R$ be a commutative ring with 1. Recall that a prime ideal of $R$ is an ideal $P$ such that

(a) $P \neq R$.

(b) For all $a$ and $b$, $ab \in P$ implies $a \in P$ or $b \in P$.

By considering the least $n$ such that $r^n \in P$ or otherwise, show that if $r$ is nilpotent and $P$ is prime then $r \in P$.

(3) Let $R$ be a commutative ring with 1, then a set $S \subseteq R$ is multiplicatively closed iff $1 \in S$, and $S$ is closed under $\times$.

Given a multiplicatively closed set $S$ such that $0 \notin S$, define $\mathcal{X}$ to be the set of ideals $I$ such that $I \cap S = \emptyset$.

(a) Show that $\mathcal{X}$ is not empty.

(b) Show that the union of a chain of elements of $\mathcal{X}$ is an element of $\mathcal{X}$.

(c) Show that any maximal element of $\mathcal{X}$ is a prime ideal of $R$.

Use Zorn’s Lemma to show that if $r \in R$ is not nilpotent, then $r \notin P$ for some prime ideal $P$.

(4) The ring $\mathbb{H}$ of real quaternions is defined as follows: the underlying set is $\mathbb{R}^4$, and by convention we write the element $(a, b, c, d)$ as “$a + bi + cj + dk$”, so that the elements $1, i, j, k$ of $\mathbb{H}$ form the standard basis of $\mathbb{R}^4$. At the risk of some confusion we sometimes identify the element $(a, 0, 0, 0)$ of $\mathbb{H}$ with the real number $a$. 


The addition is the usual addition in \( \mathbb{R}^4 \). The multiplication is defined by the following conditions:

(a) 1 is the identity.
(b) Multiplication is \( \mathbb{R} \)-bilinear.
(c) \( i^2 = j^2 = k^2 = -1 \).
(d) \( ij = k, \; jk = i, \; ki = j \).
(e) \( ji = -k, \; kj = -i, \; ik = -j \).

This may be clarified by an example:

\[
(1 + 2i + 3j)(3 + 2k) = 3 + 2k + 6i + 4ik + 9j + 6jk = 3 + 2k + 6i - 4j + 9j + 6i = 3 + 12i + 5j + 2k.
\]

It can be shown that \( \mathbb{H} \) is a division ring.

(a) Find the centre of \( \mathbb{H} \), that is to say the set of \( a \in \mathbb{H} \) such that \( ab = ba \) for all \( b \in \mathbb{H} \). Labour saving hint: find a small set of \( b \)'s such that \( a \) is in the centre iff it commutes with everything in the small set.
(b) How many solutions has the equation \( x^2 = -1 \) in \( \mathbb{H} \)?
(c) Consider the linear map \( x \mapsto (a + bi + cj + dk)x \) from \( \mathbb{H} \) to \( \mathbb{H} \).

Write down the matrix of this transformation with respect to the basis \( \{1, i, j, k\} \). What are the trace and determinant?
(d) Use the last part to give an isomorphism between \( \mathbb{H} \) and a subring of \( \text{Mat}_4(\mathbb{R}) \) (the ring of \( 4 \times 4 \) matrices with real entries).
(e) It is easy to see that the subset of \( \mathbb{H} \) consisting of elements \( a + bi \) forms a ring isomorphic to \( \mathbb{C} \). We usually identify elements of this form with the corresponding complex numbers.

Show that \( \mathbb{H} \) is a vector space over \( \mathbb{C} \) if we define scalar multiplication by \( zh = z \times_\mathbb{H} h \). Show further that \( \{1, j\} \) is a basis.
(f) Find the matrix of \( x \mapsto (a + bi + cj + dk)x \) with respect to the basis \( \{1, j\} \), and use your answer to give an isomorphism between \( \mathbb{H} \) and a subring of \( \text{Mat}_2(\mathbb{C}) \).

(5) Let \( k \) be a field and let \( R = \text{Mat}_2(k) \) be the ring of \( 2 \times 2 \) matrices with entries in \( k \).

(a) Find the centre of \( R \).
(b) Show that \( R \) is simple, but that it has a left ideal which is neither \( \{0\} \) nor \( R \).
(c) Show that \( R \) is left Noetherian and left Artinian. Hint: linear algebra ideas may help!
(d) Find \( J(R) \).
(e) Show that there are nilpotent elements not lying in \( J(R) \).

Optional and not for credit: generalise to \( \text{Mat}_n(k) \).

(6) Is \( \text{Mat}_2(\mathbb{H}) \) simple?