HW 2

JC

Due by start of class time on Fri 2. Submit it in LATEX by email to Yimu Yin (yimuy@andrew.cmu.edu)

(1) Let $G$ be a group and $X$ be a set. An action of $G$ on $X$ is transitive if and only if $X$ forms a single orbit, equivalently for all $x, y \in X$ there is $g \in G$ such that $gx = y$.

Suppose that we are given actions of some group $G$ on sets $X$ and $X'$. The actions are equivalent iff there is a bijection $\alpha : X \rightarrow X'$ such that $\alpha(gx) = g\alpha(x)$ for all $g, x$.

(a) Let $G$ act transitively on $X$, let $x \in X$ and let $H = \text{Stab}(x)$. Let $X'$ be the set of left cosets of $H$. Show that the action of $G$ on $X$ is equivalent to the action of $G$ on $X'$ by left multiplication.

We know that the orbit of $x$ is $X$, so the orbit-stabiliser idea gives a bijection $\alpha$ between $X$ and $X'$ in which $\alpha : gx \mapsto gH$. Now if $y = gx$ and $g' \in G$ then $\alpha(g'y) = \alpha(g'gx) = g'gH = g'\alpha(y)$.

(b) Let $G$ act transitively on $X$. Show that if $x, y \in X$ then the stabiliser subgroups $\text{Stab}(x)$ and $\text{Stab}(y)$ are conjugate.

Let $gx = y$. Then $h \in \text{Stab}(y)$ iff $hgx = gx$ iff $g^{-1}hg \in \text{Stab}(x)$.

(c) Let $G$ act transitively on $X$, and let $H \leq G$ be such that $H$ also acts transitively on $X$. Show that for any $x \in X$, $G = H\text{Stab}(x)$.

Let $g \in G$. Since $H$ acts transitively there is $h \in H$ such that $gx = hx$, then $h^{-1}g \in \text{Stab}(x)$ and $g \in H\text{Stab}(x)$.

(d) Let $G$ act transitively on $X$ and define an action of $G$ on $X^2$ by $g(x, y) = (gx, gy)$. Show that if $|X| > 1$ then the action of $G$ on $X^2$ is not transitive.

The “diagonal” $\Delta = \{(x, x) : x \in X\}$ is an orbit which is not all of $X^2$.

(2) Let $G$ act transitively on $X$ and let $F$ be a function with domain $X$. We say that the function is $G$-invariant iff $F(x) = F(y) \implies F(gx) = F(gy)$ for all $x, y \in X$. The action is said to be primitive if and only if every $G$-invariant $F$ is either constant or 1-1.

For any group $G$ a subgroup $H$ is maximal iff $H < G$, and there is no subgroup $K$ with $G < K < H$ (that is to say $H$ is maximal among proper subgroups of $G$).

Let $G$ act transitively on $X$ with $|X| > 1$, and let $x \in X$. Show that the action is primitive iff $\text{Stab}(x)$ is a maximal subgroup of $G$.

Note that $\text{Stab}(x) \neq G$.

Suppose first that the action is primitive and let $\text{Stab}(x) \leq K \leq H$. We have a bijection between $X$ and the left cosets of $\text{Stab}(x)$ in which $g\text{Stab}(x)$
corresponds to $gx$, and so since $\text{Stab}(x) \leq K$ the map $F : gx \mapsto gK$ is well-defined and is easily seen to be invariant. If it is constant then $K = G$ and if it is 1-1 then $K = \text{Stab}(x)$.

Conversely let $\text{Stab}(x)$ be maximal. Taking the hint we consider $K = \{g : F(x) = F(gx)\}$. By invariance we see that $K \leq G$, clearly $\text{Stab}(x) \leq K$, so $K = \text{Stab}(x)$ and $F$ is 1-1 or $K = G$ and $F$ is constant.

(3) Let $G$ be finite, let $H < G$ and let $p$ be a prime dividing $|H|$. 
(a) Show that if $P$ is any Sylow $p$-subgroup of $H$, then $G = HN_G(P)$. Hint: an earlier question may help.

For any $g \in G$, $P^g \leq H^g = H$, so $G$ acts on the Sylow $p$-subgroups of $H$. $H$ acts transitively on Sylow $p$-subgroups of $H$ by Sylow’s theorem, now use Q1.

Cultural fact: this is called the Frattini argument.

(b) Show that if $Q$ is a Sylow $p$-subgroup of $G$ then $Q \cap H$ is a Sylow $p$-subgroup of $H$.

$QH \leq G$ and $|QH| = \frac{|Q| \times |H|}{|Q \cap H|}$. Now $|QH|$ and $|Q|$ contain $p$ to the same power, so also must $|H|$ and $|Q \cap H|$.

(4) Recall that $G$ is simple iff the only normal subgroups of $G$ are $\{e\}$ and $G$. Let $G$ be a finite simple group, let $H < G$ and consider the action of $G$ on the set $X$ of left cosets of $H$ by multiplication from the left. Show that this action is an injective map from $G$ to $\Sigma_X$, and deduce that $|G| \leq |G : H|!$. Hint: HW1.

By old IW if $N$ is the kernel then $N \leq H < G$, so by simplicity $N = \{e\}$. So we have an injection into the symmetric group on the cosets which has $|G : H|!$ elements.

(5) Let $G$ be a simple group of order 60, and for $p = 2, 3, 5$ let $n_p$ be the number of Sylow $p$-subgroups. Find $n_p$.

$n_3$ divides 20 and is congruent to 1 mod 3. $n_3 > 1$ by simplicity. By the last question 4 is ruled out, and 7 is no good, so $n_3 = 10$. In particular we have 20 elements of order 3.

$n_5$ divides 12 and is congruent to 1 mod 5. $n_5 > 1$ by simplicity so $n_5 = 6$. So we have 24 elements of order 5.

$n_2$ is an odd factor of 15. $n_2 \geq 5$ by the last question so $n_2 = 5$ or $n_2 = 15$.

We know that each Sylow 5-subgroup has normaliser of index $n_5 = 6$, that is of order 10 (conversely each subgroup of order 10 has a normal subgroup of order 5, of which it must be the normaliser). So there are 6 subgroups of order 10, each the normaliser of a subgroup of order 5.

Suppose that $H$ and $H'$ are distinct subgroups of order 10, then their intersection $H \cap H'$ is a proper subgroup so has order 1, 2 or 5. 5 is ruled out because they can’t both be the normaliser of a common subgroup of order 5; 1 is ruled out because in that case we have $|HH'| = 100 > |G|$. Hence they intersect in a subgroup of order 2.

Now let $X$ be the set of subgroups of order 10 and let $G$ act on it by conjugation. Let $H \in X$, then if $g$ is in the kernel (N say) of this action we have $H^g = H$, so that $g \in N_G(H) = H$. Since $N \leq H < G$, $N$ is trivial. So we have an IM between $G$ and a subgroup of $\Sigma_X$, and enumerating
elements of $X$ as $H_i$ for $1 \leq i \leq 6$ we may set up an IM between $G$ and a
subgroup $\bar{G}$ of $S_6$.

Since $A_6 < S_6$, $\bar{G} \cap A_6 < G$, so has order 1 or 60. The order 1 case makes
$|\bar{G}A_6| = 60 \times 360 > 720$, so $\bar{G} \leq A_6$. For each $i$, conjugation by elements
of $H_i$ fixes $H_i$; so the image of $H_i$ under the IM from $G$ to $\bar{G}$ consists of
permutations which fix $i$.

Now if $i \neq j$ then the images of $H_i$ and $H_j$ intersect in $\{e, \sigma\}$ where $\sigma$
has order two and fixes $i, j$ (hence is a product of 2 disjoint 2-cycles). The
15 choices for $i$ and $j$ give us 15 distinct elements of $\bar{G}$, which must then
correspond to 15 elements of $G$. Noting that $1 + 20 + 24 + 15 = 60$, we see
all elements have order 1, 2, 3, 5. In particular the Sylow 2-subgroups of $G$
are of form $C^2_2$, where $C_2$ is cyclic of order 2; each has 3 elements of order 2,
which all commute.

Finally let $S$ be the set of elements of $A_6$ which are products of 2 disjoint
2-cycles. Easily each element of $S$ commutes with only two other elements.
So each element of order 2 in $G$ lives in a unique subgroup of order 4, and we
have that $n_2 = 5$. Now that it is too late to help us, we can easily
show that $G \cong A_6$ by letting $G$ act on cosets of the normaliser of a Sylow
2-subgroup.

A better solution to the $n_2$ part from George Schaeffer.

Sylow’s Theorem implies that $n_2 \in \{1, 3, 5, 15\}$. We can immediately
discard $n_2 = 1$ and $n_2 = 3$: the former because $G$ is normal and the
latter because then the normalizer of a Sylow 2-subgroup would have size
$60/3 = 20 > 12$ violating the conclusions drawn from Problem 4.

Suppose for now that $n_2 = 15$. The Sylow 2-subgroups of $G$ are of
size 4 and none of them can contain any of the $24 + 20 = 44$ elements of
order 3 or 5. It follows that there are distinct Sylow 2-subgroups $P, Q$ of $G$
such that there exists $g \in P \cap Q$ and $g \neq e$. Because $P, Q$ are abelian,
$P, Q \leq N_G(g)$ and so $PQ \leq N_G(g)$. Since $|PQ| = 8$, $|N_G(p)| \geq 8$. By
Lagrange’s Theorem $|N_G(g)| | 60$ so either $|N_G(g)| = 12$ or $|N_G(g)| = 60$
(remember that $|H| \leq 12$ for all proper subgroups $H < G$).

The latter possibility may be ignored because then $(g)$ is normal, non-
trivial, and proper in $G$ which violates the fact that $G$ is simple. Thus
$N_G(g)$ is a subgroup of $G$ of order 12. By Problem 4 there is an injective
homomorphism $G \to S_5$ and so $G$ is isomorphic to a subgroup of $S_5$ of
index 2 (since $|S_5| = 120 = 2|G|$); it follows that $G \cong A_5$, but this is a
contradiction because $A_5$ has only 5 Sylow 2-subgroups.

Thus $n_2 \neq 15$ so $n_2 = 5$.

(6) (You only have to do this one if you know about categories. Anyone who
took Math Studies or Commutative Algebra with me will be considered to
know about categories.)

Recall that if $A$ and $B$ are objects then an isomorphism from $A$ to $B$
is $f : A \to B$ such that for some (necessarily unique) $g : B \to A$ we have
$fg = id_B$ and $gf = id_A$. We write $g = f^{-1}$ in this case.

If $\mathcal{C}$ is a category and $X$ is an object of $\mathcal{C}$, then $Aut(X)$ is the set of all
isomorphisms from $X$ to $X$.

(a) Prove that $Aut(X)$ forms a group under composition.
(b) Prove that if $\alpha : A \to B$ is an isomorphism in $C$, then $\beta \mapsto \alpha \beta \alpha^{-1}$ is an isomorphism from $\text{Aut}(A)$ to $\text{Aut}(B)$.

Easy!